

Geometry, Topology and String Theory

by

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Abstract

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A variety of scenarios are considered which shed light upon the uses and limitations of classical geometric and topological notions in string theory. The primary focus is on situations in which D-brane or string probes of a given classical space-time see the geometry quite differently than one might naively expect. In particular, situations in which extra dimensions, non-commutative geometries as well as other non-local structures emerge are explored in detail. Further, a preliminary exploration of such issues in Lorentzian space-times with non-trivial causal structures within string theory is initiated.

Professor Bruno Zumino
Dissertation Committee Chair

I dedicate this work to Amma and Appa.

Contents

List of Figures	v
1 Introduction	1
1.1 Geometry and Perturbative Quantum Gravity	1
1.2 String Theory as Quantum Gravity and Geometry	2
1.3 Overview	5
2 D-Branes and SUSY Gauge Theories	8
2.1 Gauge Theories from String Theory	10
2.1.1 Type IIA fields and branes	10
2.1.2 IIA and M-Theory	12
2.1.3 Brane construction of $\mathcal{N} = 2$ SQCD in IIA	14
2.1.4 Little strings, renormalization, and IR-free theories	17
2.1.5 Quantum effects from branes in IIA	20
2.1.6 Quantum gauge theory and M-Theory	21
2.1.7 Soft breaking to $\mathcal{N} = 1$ SQCD	26
2.2 Summary of Field Theory Results	28
2.3 Seiberg Duality from M-Theory	30
2.4 Flavor Symmetry Breaking	34
2.4.1 Flavored magnetic monopoles	34
2.4.2 Symmetry breaking pattern	37
2.4.3 Semiclassical analysis	37
2.4.4 Nonbaryonic branches	38
2.4.5 Baryonic branch	41
2.4.6 An example, $SU(3)$ with $N_f = 4$	43
2.5 Discussion	46
3 Noncommutative Gauge Theories and the Seiberg-Witten Map	48

3.1	General Review of NCSYM in String Theory	50
3.1.1	Definition of NCYM	50
3.1.2	NCSYM from String Theory	51
3.1.3	Formal Properties of the SW Map	57
3.2	The Wess Formalism for the SW Map	60
3.3	The Ghost Field and the Coboundary Operator	62
3.4	The Homotopy Operator	65
3.5	Some Calculations	72
3.5.1	Explicit expansions of Weyl-Moyal product	75
4	Dipole Field Theories and Plane Wave Spacetimes	77
4.1	Introduction	78
4.2	Definition of Dipole Theories	80
4.3	Lightlike Dipole Theories and NSNS Plane-Waves	83
4.3.1	Geometric engineering of dipole-theories	83
4.3.2	Lightcone string theory in the NSNS background	85
4.3.3	Lightlike dipole-theories on D-branes in a plane-wave background	86
4.4	Proposal for the S-dual Theory	88
4.5	Conclusion and Discussion	93
5	Holography and the Gödel Universe in String Theory	96
5.1	Introduction and Summary	97
5.2	Holography in Gödel's Four-Dimensional Universe	102
5.2.1	Gödel's solution	102
5.2.2	Preferred holographic screens in Gödel's universe	104
5.2.3	Gödel's universe as deformed AdS_3 and holography	110
5.3	Holography in Supersymmetric Gödel Universes	112
5.3.1	Holography in the Gödel universe of $N = 1$ $d = 5$ supergravity	112
5.3.2	Preferred holographic screens	115
5.3.3	Geodesics in the Gödel universe \mathcal{G}_5	118
5.3.4	The Gödel universe of M-theory	123
5.4	Analogies with Holography in de Sitter Space	126
5.5	T-Duality of Gödel Universes	128
5.5.1	T-duality to a supersymmetric Type IIB pp-wave	129
5.5.2	Gödel/pp-wave T-duality	130
5.5.3	New supersymmetric Gödel universes in string and M-theory	131
5.6	Geometry of the Gödel Universes	132
5.6.1	The five-dimensional Gödel universe	132
5.6.2	The nine-dimensional Gödel universe	134

5.7 Discussion	134
Bibliography	136

List of Figures

2.1	Type IIA string theory realization of $\mathcal{N} = 2$ $SU(4)$ SQCD with 3 flavors. The effective worldvolume of the gauge theory (which lives on the D4 color branes) is spanned by the x^0 , x^1 , x^2 , and x^3 directions.	15
2.2	Type IIA brane configurations corresponding to the a) Coulomb branch b) a mixed Coulomb and Higgs branch.	17
2.3	IIA realization of the Coulomb branch of $\mathcal{N} = 2$ $SU(3)$ SQCD with 3 flavors	23
2.4	The M-theory lift of the above IIA configuration to a single M5-brane. The directions x^4 , x^5 , and x^6 are shown explicitly while the x^{10} coordinate is parameterized by darkness.	23
2.5	a) a baryon which wraps 2 color and 2 flavor branes in $SU(2)$ SQCD with 2 flavors (b) a meson which wraps 2 flavor branes and wraps and unwraps one color brane	25
2.6	a) r color branes and flavor brane connects. b) After rotating the five brane, color branes which are not connected to flavor branes move to the origin of v . This is the realization of the r vacua in [24].	27
2.7	An M5-brane configuration corresponding to $\mathcal{N} = 2$ $SU(3)$ SYM near a vacuum which survives upon softly breaking to $\mathcal{N} = 1$. The two small holes are the degenerating cycles. The M2-branes corresponding to nearly massless monopoles are disks bounded by these cycles. The M-theory direction is parameterized by darkness.	28
2.8	In IIA Seiberg duality is realized by brane exchange.	31
2.9	At the root of the baryonic branch, the low energy physics (to the right of the intersection) is the $SU(\tilde{N}_c) \times U(1)^{N_c - \tilde{N}_c}$ magnetic theory.	33
2.10	Baryonic root of $SU(8)$ with 13 flavors	34
2.11	a) A magnetic monopole in an antisymmetric tensor flavor representation. b) In the dual magnetic picture it becomes a baryon.	36

2.12	(a) Semiclassically when SUSY is broken by rotating the NS5 $_{\theta}$ brane, r color branes connect to flavor branes while the rest slide to $v = 0$ (b) The dual magnetic description of the nonbaryonic branches can be understood similarly to the semiclassical case.	37
2.13	This M5-brane configuration corresponds to the $r = 2$ nonbaryonic root of $SU(3)$ with 4 flavors. In the IR (the right side of the picture) a reduction to IIA produces two parallel, almost coincident NS5-branes indicating that the theory is strongly coupled. The fact that the distance between the branes converges indicates that the IR theory is superconformal.	39
2.14	Above are M5-brane cross-sections at constant x^6 corresponding roughly to constant energy scales of the $r = 2$ nonbaryonic root of the $SU(3)$ theory with 4 flavors and no bare quark masses. Reading from left to right and top to bottom the energy scale of the corresponding M2-brane probes decreases. The M theory direction is parameterized by darkness.	40
2.15	On the baryonic branch, the baryon vev $\mu\Lambda$ is realized as a relative slide of the two halves of the configuration along a collection of D6-branes.	41
2.16	The M5-brane configurations corresponding to r -vacua in $SU(3)$ with 4 flavors in the semiclassical limit, i.e. with $m_i = m \gg \Lambda$. The flavor branes and color branes which have connected are represented by semi-infinite tubes extending from the left branch of the M5-brane. The connections between the two NS5-branes are the remaining color branes, pairs of which have condensed massless monopoles between them.	44
2.17	M5-brane configurations at the r vacua in $SU(3)$ with 4 flavors and $\forall m_i = 0$.	45
2.18	Cross sections at constant x^6 of the M5-brane of Fig. 2.17b (the $r=0$ root).	45
2.19	Cross sections at constant x^6 of the M5-brane of Fig. 2.17a (the baryonic root).	46
4.1	Scattering amplitude in the lightcone formalism.	87
5.1	The geometry of the three-dimensional part \mathcal{G}_3 of Gödel's universe, with the flat fourth dimension z suppressed. Null geodesics emitted from the origin P follow a spiral trajectory, reach the velocity-of-light surface at the critical distance r_0 , and spiral back to the origin in finite affine parameter. The curve C of constant $r > r_0$ tangent to $\partial/\partial\phi$ is an example of a closed timelike curve. A more detailed version of this picture appears in Hawking and Ellis [1]. . .	104

- 5.2 The geometry of our preferred holographic screen in Gödel's universe, as defined by the inertial observer following the comoving geodesic at the origin of spatial coordinates. The translationally invariant dimension z is again suppressed. Two closed timelike curves are indicated: One, C , at constant value of $t = 0$ and $r > r_0$ is outside of the preferred screen, while another, C' , passes through the origin at $t = 0$ and intersects the screen in two, causally connected points. 106
- 5.3 The Bousso diagram for the \mathcal{G}_3 part of the Gödel universe metric, with the angular coordinate ϕ suppressed, and the structure of non-expanding light-sheets indicated by the bold wedges. The preferred holographic screen is at the finite value $r_{\mathcal{H}}$ of the radial coordinate r , strictly smaller than the location of the velocity-of-light surface at r_0 . A null geodesic sent from P would reach the velocity-of-light surface at P'' in a finite affine parameter, and refocus again at the spatial origin in P' 107
- 5.4 The behavior of null geodesics emitted from an arbitrary point P in the Gödel universe, with the initial momentum in the (x_1, x_2) plane, and with several such geodesics indicated. Each geodesic travels along a circular trajectory, reaches the velocity-of-light surface and returns back to P , penetrating the preferred screen exactly twice during each rotation cycle. 122
- 5.5 The two types of preferred screens in the M-theory Gödel $\mathcal{G}_5 \times \mathbf{R}^6$. The translationally-invariant screen is located at $r_{\mathcal{H}}$ in \mathcal{G}_5 for all values of $|y|$, and can be associated with an extended observer delocalized or wrapped along y_a . The screen associated with a localized observer is compact in all space directions, and extends beyond $r_{\mathcal{H}}$, closer to the velocity-of-light surface r_0 125

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Chapter 1

Introduction

1.1 Geometry and Perturbative Quantum Gravity

Geometric concepts play a central role in the construction of models for physical phenomena. It is remarkable that the geometric model of space-time as a smooth manifold endowed with a Lorentzian metric (see, for example, [1, 2, 3]) is applicable to the description of physics ranging from the subatomic to the astronomical. Indeed, it is difficult to imagine a description of physics which does not involve the data of relative temporal and spatial distances and angles between objects.¹ The highly successful local, Lorentz invariant and renormalizable quantum field theories (QFT's) used to construct the Standard Model of particle physics certainly require such classical data as inputs. As Einstein taught us, gravity is associated with the dynamics of the space-time metric itself. Quantizing gravity then seems to require reconciling the dependence of the quantization procedure on the classical expectation value of the metric with the requirement that the metric be treated as a fully quantum field. This situation is analogous to that encountered for a Higgs field, and suggests quantizing small perturbations about a given background configuration.² Unfortunately, unlike

¹Yet, such theories do exist and are of significant mathematical interest - topological quantum field theories and string theories. However, being devoid of dynamics, they seem to be of little physical interest.

²See [8] for details regarding an attempt to quantize gravity in a manifestly background independent way.

the case of the Higgs theory, the usual methods of perturbative QFT applied to fluctuations of the metric lead to a non-renormalizable theory. In particular, just as was the case for the Fermi theory of the weak interactions, the strength of gravity is controlled by a coupling G_N with inverse squared mass dimension leading to uncontrollable divergences in loop corrections coming from high energies or small distances. More intuitively these divergences arise due to the fact that probing short distances, $\Delta x \approx \ell_p = 1.6 \times 10^{-33} \text{ cm}$ requires, by the usual arguments of quantum mechanics, very high energy probes $E \approx 1/\ell_p = 1.22 \times 10^{19} \text{ GeV}$ which lead to large local backreaction with induced radius of curvature, $R \approx \ell_p$.

In the case of the Fermi theory, these divergences were of course later understood to be the signal of new physics at energy scales near where the effective dimensionless coupling $G_F E^2 \approx 1$. At these energy scales, the interaction vertices between four fermions in the Fermi theory are actually understood to be split into a pair of trivalent vertices connected through the exchange of virtual W^\pm and Z^0 gauge bosons whose couplings and masses determine G_F . So the interaction vertices of the Fermi theory get smeared out by the propagation of virtual gauge bosons, thereby helping to soften the small distance divergences in the Fermi theory. In fact, using in a crucial way the hidden gauge symmetry, it was shown that the above procedure results in a local, renormalizable description of the weak interactions.

In the seventies and eighties, it was recognized that perturbative superstring theory potentially provided a framework for an analogous explanation for the divergences of perturbative quantum gravity (see [4, 5] for a review). The excitement about string theory in recent years can be attributed to the realization that superstring theories are actually *non-perturbatively* consistent quantum theories containing both Einstein's gravity and supersymmetric QFTs (see [6, 7] for a review of these newer developments).

1.2 String Theory as Quantum Gravity and Geometry

Just as in the case of the weak interactions, string theory introduces new degrees of freedom which would appear when $G_N E^2 \approx 1$ whose propagation and gauge symmetries

smear out gravitational vertices and in fact render gravitational amplitudes *finite*, order by order in a perturbative expansion scheme.³ However, string theory achieves this by smearing out both the interaction vertices as well as the fundamental degrees of freedom themselves. Further, the consistency of the expansion scheme actually requires that *all* the perturbative degrees of freedom (not just gravitational) of the theory be associated with open or closed strings. In particular, the graviton is interpreted as one of the lowest excited states of a closed superstring, while gauge bosons and other matter arise as other states of closed or open strings. That is, string theory is no longer manifestly *local*, and in fact does not contain truly local perturbative probes.⁴ The absence of such local probes calls into question the interpretation of the classical background metric associated with the theory at distance scales smaller than the characteristic length of the string, $\sqrt{\alpha'}$. Note that this issue exists even in the “classical limit” of the string theory, by which we mean the limit in which the expansion parameter g_s is set to zero, but where the non-locality remains.

In order to approach this issue, it is useful to briefly recall the way in which the classical, background geometric data enters into string theory in this “classical limit”. A classical vacuum state of a closed superstring theory is specified by the choice of a superconformal 2D theory.⁵ Generically, these CFTs can be understood as conformally invariant non-linear sigma models describing the embedding of the string world-sheet into a geometric target space-time. The couplings of the non-linear sigma model are interpreted as specifying the background metric, string coupling, as well as certain higher form field strength fluxes in the target space-time. The superconformal symmetry places significant constraints on these couplings, which at lowest order in a systematic expansion in α' require that the classical background fields obey generalizations of Einstein’s equations. In this way, a choice

³Note, however, that the string perturbation expansion does not converge! In fact, it is not even Borel summable,[9] suggesting the presence of non-perturbative (non-analytic in the expansion parameter) corrections of great importance. In fact, such corrections have been understood as effects associated with D-branes[12, 13], and even computed using certain dualities.

⁴This is true only perturbatively - D0-brane probes in IIA provide a non-perturbative counterexample of great importance.

⁵We are ignoring here an important subtlety - such a formulation is not fully understood in the presence of RR fluxes (see [10] for the state of the art regarding this issue).

of vacuum in string theory corresponds to a particular Lorentzian space-time manifold with certain additional fluxes and solitonic defects in which the string is interpreted as propagating. However, this correspondence is quite subtle. First, the space-time manifold may be singular without spoiling the consistency of the theory, which suggests that in some sense string theory resolves the singularity of the classical geometry.⁶ Further, the mapping of vacuum data to geometry is not unique, as the CFTs associated with different manifolds can be isomorphic. Strings propagating in certain compact space-time manifolds with a certain characteristic size R can be re-interpreted in terms of a string theory on an often topologically distinct manifold of size $\frac{\sqrt{\alpha'}}{R}$. This duality involves the exchange of string winding and momentum modes around various circles [15] and so makes explicit use of the non-locality of the theory. Thus, we see that string theory in a given vacuum configuration may admit more than one semiclassical, geometric interpretation, suggesting that stringy probes experience the classical geometry in a more sophisticated way than is the case in QFT.

Things get much more interesting when we go beyond the consideration of perturbative string probes present in the classical ($g_s = 0$) limit of string theory. One of the major events that led to the recent renaissance in string theory was the realization of the importance of additional probes in quantum string theory, D-branes. D-branes in type I and II string theories are extended objects which are charged under the antisymmetric tensor fields coming from the RR sector of the superstring. At small g_s , their fluctuations are described by open strings restricted to end upon them [12, 13, 14]. At low energies, the lightest modes of these open strings are just gauge bosons and their superpartners, giving rise to an effective description of the physics of N coincident D-branes in terms of an $SU(N)$ super Yang-Mills (SYM) theory localized on their world-volume. Further, we note that an alternate description of these Dp-branes, as extremal p-brane solitons of the effective low energy supergravity theory, becomes valid when $g_s N$ is large. Note also that the tensions of these

⁶Interestingly, one can trace the origin of this resolution to the fact that certain NSNS B fluxes are present in the geometry [11]. This suggests that such fluxes ought to be more intrinsically included in any effective geometry experienced by a string probe. In fact, as we will discuss in Chapter 3, this issue plays an important role in understanding how noncommutative geometry enters into string theory.

D-branes scale as $\frac{1}{g_s}$, suggesting that D-branes wrapped on compact cycles can become light at strong string coupling. Thus, there are many interesting circumstances in string theory where the open strings ending on these branes are the most interesting degrees of freedom, or even where the D-branes themselves become very light and dominate the physics. In these circumstances, it is the effective geometry associated with semiclassical physics of these probes that becomes the most physically relevant.

1.3 Overview

The work described in this dissertation primarily focuses on analyzing circumstances in string theory where the effective geometry associated with string or D-brane probes differs significantly from the background metric associated with the couplings of the worldsheet non-linear sigma model. In each case, interesting physical and mathematical consequences of these differences are explored. It is hoped that these inquiries will help in approaching some of the difficult issues as yet unresolved in relating string theory to natural phenomena such as vacuum selection, the cosmological constant problem and supersymmetry breaking.

In Chapter 2 we begin with an example in which the duality between a string theory and 11D supergravity (M-theory) can be used to gain a geometric understanding of a variety of non-perturbative phenomena in supersymmetric gauge theories [44]. As we mentioned earlier, the low energy effective description of the open string theory associated with N coincident D-branes is $SU(N)$ SYM theory. In fact, one can realize supersymmetric analogues of 4D QCD within string theory using brane constructions [21, 22]. Using the novel geometry associated with the physics of D0- and D2-brane probes of this brane construction and their lift to M-theory, we are able to gain a better understanding of the mechanism of flavor symmetry breaking by flavored monopole condensation suggested in the work of [43] in these theories. Further, by a novel re-interpretation of the relation between the lifts of IIA brane configurations to M-Theory, we were also able to give a simple interpretation of Seiberg Duality [17] in this framework.

The gauge symmetry on the world-volume of D-branes has a further important conse-

quence in that the fields which describe the fluctuations of the D-brane are actually adjoint valued, $N \times N$ matrices. In this sense, the “coordinates” of these coincident D-branes have become noncommuting matrices. This hints at a role for noncommutative geometry in the description of the fluctuations of D-branes [20]. As we will review in Chapter 3, the easiest way to understand the emergence of noncommutative geometry is to add a constant NSNS antisymmetric tensor flux $B_{\mu\nu}$ along the world-volume of the branes (the connection to matrices will be made clear later). The effective physics can be described in terms of an ordinary gauge theory in the presence of constant magnetic flux. However, in a certain limit involving large B the physics of fluctuations of the D-brane are best described by a non-local theory with a deformed noncommutative gauge invariance defined on a noncommutative deformation of space-time [58, 59],

$$[x_\mu, x_\nu] \approx \theta_{\mu\nu} \tag{1.1}$$

where $\theta \approx \frac{1}{B}$. Seiberg and Witten [59] argued that there must be a mapping between these descriptions, and proposed a form of this map for the $U(N)$ case to first order in θ . In Chapter 3 we discuss this correspondence in greater detail and discuss the formal properties of this map [74]. We find a cohomological formulation of the Seiberg-Witten map using BRST methods. Using this interpretation, we present an effective method for computing the map order by order in θ for the general, non-abelian case.

In Chapter 4, we study a simpler non-local deformation of SYM known as a dipole deformation [77] which result if one places branes in a strong NSNS field-strength. The fields of such a theory couple non-locally to the world-volume gauge theory with opposite charges at some space-time separation, i.e. as dipoles. In particular, we found that D-branes in the Penrose limit of $AdS_3 \times S^3 \times T^4$ can be described by a light-like dipole theory [16].

Perhaps the most radical proposal for how at tiny distance scales to our notions of space-time is the idea of Holography as expounded by 't Hooft and Susskind. Roughly, this is the notion that the true degrees of freedom of a gravitational theory in $d + 1$ dimensions are radically fewer than one might naively expect. This idea motivated by studies of black hole thermodynamics in classical general relativity which suggest that the area of the horizon of a

black hole acts in every way like, and must be treated as, an entropy. Now, as gravitational collapse can only lead to increasing entropy density, a black hole represents a state of maximal entropy density. Quantum mechanically, we expect that entropy counts the number of accessible quantum states of a system. Thus, we are led to expect that the degrees of freedom in a quantum mechanical, gravitational system ought to scale with area rather than with volume. Bousso (building on earlier work by Fischler and Susskind) formulated this conjecture more precisely for arbitrary space times as the covariant entropy bound, which states that the entropy flux through any given light-sheet in space-time is bounded by one quarter its area in Planck units. Further, he established a precise criterion for finding certain hypersurfaces in spacetime which bound light sheets of maximal possible entropy called preferred holographic screens.

This suggests that perhaps the physics of a gravitational system in $d + 1$ dimensions may be described by a non-gravitational theory in one lower dimension which may, in some sense, “live on the holographic screen”. In fact, a conjectural example of such a relationship, the AdS-CFT correspondence, has been explored in great detail in string theory. The most familiar version of this correspondence arises by considering N $D3$ -branes in IIB, whose low energy worldvolume theory is just the $d = 4$, $\mathcal{N} = 4$ $SU(N)$ superconformal Yang-Mills theory. Maldacena conjectured that string theory on the near horizon $AdS_5 \times S^5$ geometry of the supergravity description of these D-branes should be dual to this superconformal theory, which can be understood as “living on the boundary” of AdS_5 .

In Chapter 5, we apply Bousso’s procedure to obtain preferred holographic screens and entropy bounds in certain supersymmetric analogues [138] of a rotating universe of a kind first constructed by Gödel [137] in 1949. These spacetimes have causality problems associated with the presence of closed time-like which, we suggest, may have an interesting resolution in a holographic interpretation upon the preferred screens we computed. In order to test this proposition, we would like to analyze these backgrounds in string theory. It turns out [159, 160] that these backgrounds are T-dual to certain exactly solvable plane-wave spacetimes (incidentally, S-dual to those considered in Chapter 4).

Chapter 2

D-Branes and SUSY Gauge Theories

One of the main sources of the renaissance in string theory in the last decade of the twentieth century was an exact result in field theory. Seiberg and Witten [18] proposed an exact the low energy effective action for $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theory. This proposed “solution” of the theory made extensive use of the concepts of electric-magnetic duality as well as the holomorphicity properties of certain observables in the theory that had their roots in the unbroken supersymmetry. The solution was encoded, for reasons that were quite mysterious at the time, in terms of a complex curve, later known as the Seiberg-Witten curve. The arguments used to make this proposal were subsequently vastly generalized in string theory, where they were used to analyze the strong coupling ($g_s \rightarrow \infty$) behavior of various string theories (see e.g. [7] for details).

In this chapter, we will consider doing the opposite. Maximally supersymmetric gauge theories are realized on the worldvolumes of D-branes. As we will review in section 2.1, one can use various combinations of branes to reduce the supersymmetry of these gauge theories as well as couple matter to them. Unhappily, the analysis of these configurations in IIA involves non-perturbative, strong-coupling effects. However, using a now well established

duality between IIA string theory and M-theory (which will be reviewed briefly in section 2.1), it was shown in [22] that supersymmetric QCD (SQCD, supersymmetric Yang-Mills coupled to supersymmetric matter transforming in the fundamental representation of the gauge group) is contained in the low energy description of certain configurations of M5-branes in multi-centered Taub-NUT space [22]. It turns out that the classical properties of these brane configurations in M theory are able to provide a concrete realization of otherwise mysterious quantum phenomena, such as confinement via magnetic monopole condensation [18, 23, 49] and Douglas-Shenker strings [47, 51]. This approach is known as MQCD, and is reviewed extensively in [50]. Here, we introduce a novel MQCD description of Seiberg duality in $\mathcal{N} = 2$ superQCD softly broken to $\mathcal{N} = 1$ by adding a mass term for the adjoint chiral multiplets. We then proceed to construct the M-theory realization of the $U(N_f) \rightarrow U(r) \times U(N_f - r)$ flavor symmetry breaking mechanism in $\mathcal{N} = 1$ $SU(N_c)$ gauge theories studied in Refs. [42, 43, 54].

In section 2.1, we briefly outline the relevant field and brane content of type IIA string theory and proceed to review the semi-classical description of supersymmetric gauge theories as effective field theories of parallel D4-branes suspended between NS5-branes in type IIA string theory. To understand the quantum gauge theories we lift this description to M-theory. We then review the field theory results of Refs. [42, 43, 54] in section 2.2. Then, in section 2.3 we describe Seiberg duality as a choice of two D-brane configurations in IIA which lift to the same M5-brane configuration as seen by M2-brane probes at different energy scales. We finally turn to the MQCD description of flavor symmetry breaking in section 2.4 and present a detailed example of the M5-brane configurations corresponding to r-vacua in $SU(3)$ with 4 flavors is presented in section 2.4. Note that the results of this chapter were obtained in collaboration with Jarah Evslin, Hitoshi Murayama, and John Wang [44].

2.1 Gauge Theories from String Theory

After reviewing the basic features of type IIA string theory and M-theory we will require, we consider the embedding of $3 + 1$ dimensional $\mathcal{N} = 2$ $SU(N_c)$ SQCD with N_f flavors (i.e. N_f hypermultiplets transforming in the fundamental representation of $SU(N_c)$) in IIA string theory and its M-theory lift [21, 22].

2.1.1 Type IIA fields and branes

Let us recall the field and brane content of type IIA string theory at low energies (all the facts we state below are fairly standard and reviewed in, e.g. [7]). At low energies, the type IIA string reduces to type IIA supergravity in ten dimensions. This theory has two 16 component Majorana-Weyl supersymmetries of opposite ten dimensional chirality, for a total of 32 real supercharges. Both type IIA and IIB string theory share the same field content coming from the NS sector of the superstring - massless modes associated with fluctuations of the space-time metric $g_{\mu\nu}$, an antisymmetric two-tensor gauge field $B_{\mu\nu}$ (the NS B -field), and the dilaton Φ . The background value of the dilaton sets the string coupling constant,

$$g_s = e^{2\Phi}. \quad (2.1)$$

Fundamental strings are electrically charged under the NS B -field. There exist half-BPS (configurations preserving half of the supercharges) solitonic 5-branes called NS5-branes which are magnetically charged under the NS B -field. The background field configuration associated with a stack of k NS5-branes is given by [26],

$$\begin{aligned} e^{2(\Phi-\Phi_0)} &= 1 + \sum_{i=1}^k \frac{\alpha'}{|\vec{x}-\vec{x}_i|^2} & G_{IJ} &= e^{2(\Phi-\Phi_0)} \delta_{IJ} \\ H_{IJK} &= -\epsilon_{IJKL} \partial^L \Phi & G_{\mu\nu} &= \eta_{\mu\nu}, \end{aligned} \quad (2.2)$$

where $I, \dots, L = 6, \dots, 9$ label directions transverse to the fivebranes, \vec{x}_i are the positions of the branes in those directions, and $\mu = 0, \dots, 5$ label directions along the brane. The stability of this configuration arises of coincident NS5-branes arises from the cancellation of their mutual magnetic repulsion and gravitational attraction and is associated with the fact

that they are half-BPS. Note the divergence of the dilaton near the cores of the NS5-branes which would suggest that perturbative string theory breaks down there. The world-volume theory on the NS5-branes is a poorly understood supersymmetric theory of antisymmetric two-tensor “gauge” fields with self-dual field strength.

The IIA and IIB string theories differ in the fields that arise from the RR sector of the superstring, as well as the corresponding D-branes charged under them. In type IIA, we find even dimensional anti-symmetric tensor gauge fields C^1 , C^3 . The D0-brane is electrically charged under C^1 , while the D6-brane is magnetically charged under C_1 . Similarly, the D2-brane and D4-brane carry respectively electric and magnetic charge under C^3 . These D-branes are also half-BPS, and carry maximally supersymmetric gauge theories on their worldvolumes.

An extremely important feature of these D-branes that we will use extensively in the following is the fact that D-branes can end on, dissolve in, and intersect other D-branes and NS5-branes in various ways that preserve large fractions of supersymmetry. The fact that branes can end and dissolve is surprising at first glance as all these objects carry gauge charges which ostensibly need to be conserved. The charges of the dissolved or ending branes can be carried by the gauge fields on the branes they end on or dissolve in. The particular cases of interest for this work are the following, all of which are at least $\frac{1}{4}$ BPS configurations,

- Fundamental strings, of course, can end on any D-brane and describe the low-energy fluctuations of the D-brane at small g_s . The ends of strings are electric charges for the world-volume gauge fields on any D-brane.
- D0-branes can dissolve into gauge theory instantons on the worldvolume of D4-branes, as can D2-branes into D6-branes.
- D2-branes can end on D4-branes and NS5-branes. The end of the D2-brane is a magnetic monopole of the gauge theory on the D4-brane.
- D4-branes can end on NS5-branes. The end of the D4-brane is described as a vortex

in the NS5-brane worldvolume theory.

- D4-branes can also end on D6-branes and are magnetic monopoles on the worldvolume of the D6-branes.

2.1.2 IIA and M-Theory

We will now review the relevant facts regarding the duality of type IIA string theory and M-theory compactified on S^1 . The bosonic fields of maximally supersymmetric 11D supergravity are an antisymmetric 3-tensor field C and a metric g . The 11D supergravity theory admits half-BPS solutions, M2-branes, which are electrically charged under the C -field, as well as smooth magnetically charged solitons, M5-branes.

When M-theory is compactified on a circle of radius R , we can immediately see that the field content of the theory matches that of type IIA supergravity. The Kaluza-Klein gauge fields arising from the components of the metric and C -field along the circle yield the RR gauge field C^1 and NS B -field and respectively. The components of the C -field and metric transverse to the circle respectively yield the IIA RR C^3 -field and the IIA 10D metric. The IIA dilaton is associated with the component of the metric which corresponds to the size of the circle. This gets to the heart of the duality - as the vacuum expectation value of the dilaton determines the string coupling, we expect that the size of the circle in M-theory encodes g_s . In particular, classical supergravity is a valid description of the physics when the circle is large, which corresponds to the strong coupling limit of the string theory. The precise relation (see [6]) is that $R(g_s) \propto g_s^{2/3}$. This is the fundamental fact which we will exploit to understand non-perturbative behaviour in gauge theories living on D-branes in IIA via M-theory.

Now let try to understand how the BPS spectrum of branes and strings in IIA arise from corresponding objects in M-theory.¹ We begin with strings and D2-branes. As the fundamental string is electrically charged under the NS B -field, which arises from the KK-

¹The fact that these states are BPS is crucial to give meaning to these correspondences. The BPS condition ensures that these states, defined in IIA at small string coupling, are stable to quantum corrections and can be reliably followed from small string coupling to large string coupling to states in M-theory.

reduction of the C -field, it is clear that it must be related to an M2-brane which wraps the S^1 . Similarly, the D2-brane arises from an M2-brane whose worldvolume is transverse to the S^1 . The important point here is that the D2-brane and the string have a common origin in M-theory. An analogous relation holds between the electric-magnetic duals of these objects in IIA, the NS5-brane and D4-brane, and the M5-brane in M-theory. D4-branes arise from wrapped M5-branes while NS5-branes correspond to unwrapped M5-branes. This common origin will be essential for what follows.

The D0-brane is electrically charged under C^1 and as charge under KK-reduction is associated with momentum in the compact direction, D0-brane charge is understood to correspond to momentum along the S^1 . The fact that D0-branes become light at strong string coupling is interpreted in M-theory via the need to include the full tower of KK-excitations in the theory.² That is, the apparently uncontrollable non-perturbative physics of light D0-branes in IIA string theory at large coupling is mapped to the highly tractable, classical problem of including the effects of propagation in an extra dimension. So the physics of light D0-brane probes led us to a novel geometric re-interpretation of the physics of IIA string theory in terms of an 11D theory. This is perhaps the most important fact for gaining intuition about how M-theory simplifies and controls the strong coupling limit of IIA. The importance of D0-branes in this correspondence is a recurring theme - in fact, according to the Matrix theory conjecture of Banks and collaborators [115], there is a limit in which the physics of these D0-branes describes M-theory completely.

Now, the D6-brane is the electric-magnetic dual of the D0-brane, and must correspond to a magnetic monopole associated with the KK-gauge field. Such monopole charges must correspond to space-times where the associated KK-gauge fields have monopole-like defects. The most basic such space-times are known as “Kaluza-Klein monopoles” or multi-Taub-NUT spaces. These spaces are asymptotically flat and appear at infinity to be non-trivial S^1 bundles over \mathbb{R}^3 , the twisting of the bundle giving the Kaluza-Klein magnetic charge.

²Note that a highly non-trivial check of this conjecture is the requirement that precisely one D0-brane bound state occurs for each value of D0-brane charge and have precisely the integral masses so as to fill out a KK-tower at strong coupling! This non-trivial check has actually been carried out in many string backgrounds [20, 27].

To be more explicit, the D6-branes are associated with 11D metrics which are flat along their $\mathbb{R}^{6,1}$ world-volumes (which we take, for consistency with what follows, to be in the x^0, \dots, x^3 and x^7, \dots, x^9 directions), and have the following multi-Taub-NUT form in the remaining directions,

$$ds^2 = \frac{V}{4} d\vec{r}^2 + \frac{V^{-1}}{4} \left(\frac{dx^{10}}{R} + \vec{\omega} \cdot d\vec{r} \right), \quad V = 1 + \sum_{i=1}^k \frac{1}{|\vec{r} - \vec{r}_i|}, \quad \nabla \times \vec{\omega} = \nabla V, \quad (2.3)$$

where x^{10} is the direction upon which we are compactifying and $\vec{r} = R(x^4, x^5, x^6)$. Note that as long as multiple sixbranes do not become coincident, this metric is smooth and complete. Further, as the relative distances in the above metric scale with R , we find that D6-branes have a smooth description in M-theory as long as R is large, i.e. in the region where the supergravity description is valid.

2.1.3 Brane construction of $\mathcal{N} = 2$ SQCD in IIA

Consider type IIA string theory on $\mathbb{R}^{9,1}$ with coordinates x^0, \dots, x^9 and complex coordinates

$$v = x^4 + ix^5 \quad w = x^8 + ix^9. \quad (2.4)$$

A stack of N_c parallel D4 “color” branes extend along directions x^0, x^1, x^2, x^3 , and x^6 . The low energy theory on these branes is $SU(N_c)$ 4+1 dimensional super Yang-Mills with 16 supercharges. In order to get a theory with 8 supercharges and of the right dimension, we will take these D4-branes to stretch between the cores of pair of NS5-brane solitons extended along the x^0, \dots, x^5 directions and placed at positions 0 and L_6 along the x^6 direction. The effective gauge coupling, g , of the 3+1 dimensional theory is given by

$$\frac{1}{g^2} = \frac{L_6}{g_s l_s} \quad (2.5)$$

where g_s , l_s , and L_6 are the string coupling constant, the string length and the distance between the two NS5-branes. To decouple the degrees of freedom of the bulk from the color branes we take the limits

$$g_s \rightarrow 0, \quad \frac{L_6}{l_s} \rightarrow 0, \quad g = \text{constant}. \quad (2.6)$$

As we will explain in greater detail in the next section, this double scaling limit is highly non-trivial, as string theory near the NS5-branes is far from flat space perturbative IIA.

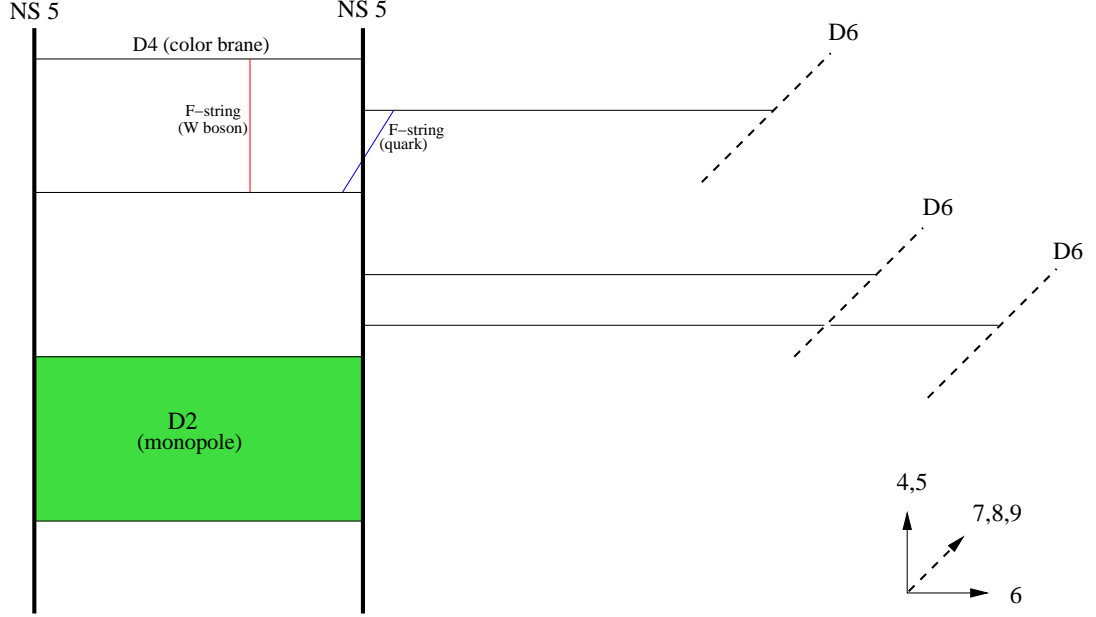


Figure 2.1: Type IIA string theory realization of $\mathcal{N} = 2$ $SU(4)$ SQCD with 3 flavors. The effective worldvolume of the gauge theory (which lives on the D4 color branes) is spanned by the x^0 , x^1 , x^2 , and x^3 directions.

The light perturbative degrees of freedom of the gauge theory are strings which stretch between the color branes, yielding an $\mathcal{N} = 2$ vector multiplet transforming in the adjoint representation of $SU(N_c)$. The distances between color branes correspond to the vacuum expectation values of the adjoint scalars in the vector multiplet and so parameterize the Coulomb branch. Quark hypermultiplets transforming in the fundamental representation of $SU(N_c)$ may be included by attaching N_f D4 “flavor” branes stretching between one of the NS5-branes and a D6 flavor brane. To preserve $\mathcal{N} = 2$ supersymmetry no two D4-branes may connect the same NS5 and D6-brane. This is known as the s-rule [21] and is U-dual [28, 29] to Pauli’s exclusion principle. The quarks and squarks are strings which stretch

from one D4 flavor brane to one D4 color brane and so they transform in the fundamental representations of both the $SU(N_c)$ gauge group and the global flavor symmetry group. Semiclassical magnetic monopole and dyon [22, 39, 40] states are realized by D2-branes with the topology of a disk bounded by a D4-NS5-D4-NS5 cycle. The brane configuration is summarized in Table 2.1.

Brane	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D4	X	X	X	X			X			
NS5	X	X	X	X	X	X				
NS5 $_{\theta}$	X	X	X	X	(X)	(X)			(X)	(X)
D6	X	X	X	X				X	X	X
D2	X				(X)	(X)	X			
D0	(X)						(X)			

Table 2.1: Alignments of branes in IIA. Parentheses indicate that the brane may be aligned at an angle between the given directions. For the D0-branes they are used to indicate the presence of both D-instantons and dynamical D0 particles.

There is an unbroken global $U(N_f)$ symmetry when the flavor branes are placed at the same v coordinate, although this symmetry is broken to $U(1)^{N_f}$ when they are placed at distinct positions $v = m_i$, $i = 1, \dots, N_f$. The m_i are the bare quark masses. Generally, a quark with flavor i and color $a = 1, \dots, N_c$ has mass

$$m_i^a = |m_i - \phi^a| \quad (2.7)$$

which is the shortest distance between color brane a and flavor brane i . Alternately the quark mass can be read from the superpotential terms:

$$W \supset \int d^2\theta \{ \tilde{Q}_i \Phi Q^i + m_i \tilde{Q}_i Q^i \} \quad (2.8)$$

where Q_i, \tilde{Q}_i are the $\mathcal{N} = 1$ chiral multiplets of the quark hypermultiplet and Φ is the $\mathcal{N} = 1$ chiral multiplet of the $\mathcal{N} = 2$ vector multiplet.

When two flavor branes i and j and a color brane a are at the same v coordinate, $\phi^a = m_i = m_j$, then it is possible to enter the Higgs branch of the gauge theory. This is

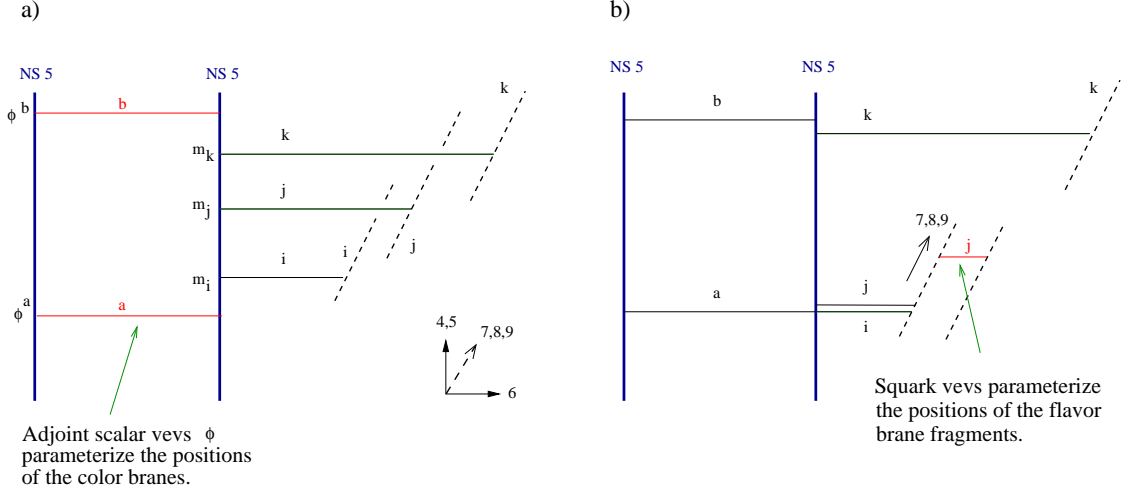


Figure 2.2: Type IIA brane configurations corresponding to the a) Coulomb branch b) a mixed Coulomb and Higgs branch.

done by connecting color brane a to flavor brane i and then breaking flavor brane j on D6-brane i . At this point we are allowed to move the portion of D4-brane j which is between D6-branes i and j , corresponding to generating vacuum expectation values for the squarks in the hypermultiplet. These vacuum expectation values are parameterized by the position of the D4-brane in the x^7 , x^8 , and x^9 directions as well as the Wilson line of the gauge field A_6 .

The $U(1)_R \times SU(2)_R$ R-symmetry of the classical $\mathcal{N} = 2$ theory is manifested as a rotational symmetry of the brane cartoon. The $U(1)_R$ symmetry corresponds to rotations of the v -plane, while the $SU(2)_R$ is the universal cover of the $SO(3)$ acting on x^7 , x^8 , and x^9 by rotations.

2.1.4 Little strings, renormalization, and IR-free theories

Since the limit we take to obtain the gauge theory is one in which $L_6 \ll l_s$, all the the above strings and branes are necessarily very near the cores of the two NS5-branes. The

closed string background fields associated with the NS5-brane solutions deviate significantly from their asymptotic flat space values in that region. For instance, the string coupling diverges near the cores of the NS5 branes. Thus, these objects are not well described by branes and perturbative strings in flat space and are better understood as bound to the NS5-branes. That is, they should be described as boundary states and their corresponding open string sectors in the CFT which describes the background associated with the NS5-branes. In particular, the masses of string states and the tensions of branes will differ from the values that one might naively guess from the cartoons drawn above.

In the double scaling limit we are taking, ($L_6, g_s \rightarrow 0$ with $L_6/g_s l_s$ held constant [30, 31]), the theory living on the NS5-brane decouples from the bulk and is a non-gravitational six dimensional string theory known as Little String Theory (LST) [32]. In [33] a proposal was made to study LST holographically by considering string theory in near horizon limit of the curved geometry near the two NS5-branes. Fortunately, this string theory is described by an exactly solvable CFT, and an analysis of branes and strings in this holographic description has actually been carried out in detail in [34]. Their analysis confirms the validity of many of the heuristic arguments we will make regarding these strings and branes in IIA.

The renormalization group flow of the coupling constant can be understood using such a simple argument. The ends of D4-branes act as charges (in fact, vortices as we will see later in M-theory) in the world-volume theory of the NS5-brane. At low energies, this theory is a mysterious six dimensional (2,0) tensor multiplet theory. Its bosonic fields are a two form gauge field along with five scalars, four of which parameterize transverse fluctuations of the brane while a fifth living on a circle foreshadows its lift to M-theory. Reducing the theory to the transverse space to the vortex, it becomes a supersymmetric 2+1 dimensional gauge theory with D4-branes acting as electric charges. The high degree of supersymmetry ensures that there is no net force between D4-branes. However, the charges will actually bend the NS5-brane by inducing the field profile $x^6 \sim \log |v|$ (related by supersymmetry to the profile of the gauge field around a charge in 2+1 dimensions). In particular, this implies

that the gauge coupling of the theory runs [22],

$$\frac{1}{g(v)^2} = \frac{L_6(v)}{g_s l_s} \sim \log |v|. \quad (2.9)$$

It is worthwhile reviewing what we mean by a running coupling in this context in a little greater detail. At a generic point on the Coulomb branch of the theory with generic quark masses, the only massless fields are the $U(1)^{N_c-1}$ vector multiplets in the Cartan subalgebra of $SU(N_c)$. Since we have no charged massless fields, we expect that below the scale associated with the lightest massive particle the gauge couplings of the $U(1)$'s do not run and can be considered fixed functions of that scale. Now, as the masses of particles in the IIA construction are related to distances between branes in v , the coupling is a function of these distances. This is what we mean when we interpret Eqn. 2.9 as a running coupling.

However, our primary interest is precisely in non-generic singular points (see section 2.2) in the moduli space of these theories where there *are* extra massless states. In these cases, the characteristic scale v vanishes, reflecting a singularity of the corresponding effective action arising from integrating out these extra massless states. Often, these states will be mutually non-local, corresponding to electrically and magnetically charged particles for the same $U(1)$, and therefore result in superconformal theories [35, 36]. If the superconformal theory is trivial (as is the case for most of the situations we will consider), the resolution to these difficulties is just to modify the effective action near these singular loci to include the extra massless degrees of freedom. Of course, if the theory includes massless monopoles which are described by non-perturbative states (massless D2-branes) in IIA, we will need better knowledge of non-perturbative stringy effects to analyze these configurations. We will later use M-theory for this purpose.

As we will discuss in section 2.3, in the case a trivial superconformal theory (an IR free gauge theory), there is an interesting IIA description that is motivated by M-theory considerations. At low energies, these theories are weakly coupled, so we can consider a IIA brane configuration such that $\frac{L_6}{l_s} \gg 1$ and $g_s \ll 1$. If $E \ll 1/L_6 \ll l_s$, then bulk gravity decouples and we find an IR free gauge theory. However, at energies greater than $1/L_6$, we expect that the theory will significantly deviate from a 4D gauge theory due to

the presence of A^6 KK modes. Thus, we can interpret this theory as a KK reduced 5D gauge theory weakly coupled to 4D matter (the quarks are still localized near the core of an NS5-brane). The weakness of the coupling is precisely due to the small overlap of the 5D gauge boson wavefunctions with the 4D quarks localized at the branes. Note that the gauge bosons are well described by perturbative IIA strings in flat space since $L_6 \gg l_s$. Further, note that the Coulomb branch of the field theory is insensitive to the x^6 position of the D6-branes to which the D4 flavor branes are attached (see Fig. 2.1). So, we are free to move the D6-branes along the x^6 direction through one of the NS5 branes (thereby undergoing Hanany-Witten transitions [21] that eliminate the flavor D4-branes) such that all the D6-branes are between the two NS5-branes and far from their cores. The quarks in this picture are perturbatively described by D4-D6 strings in flat space.

2.1.5 Quantum effects from branes in IIA

The most important feature of the IIA construction of gauge theories is that branes provide physically intuitive realizations of quantum effects due to monopoles and instantons in these gauge theories.

Semiclassical magnetic monopole and dyon [22, 39, 40] states are realized by D2-branes with the topology of a disk bounded by a D4-NS5-D4-NS5 cycle. The mass of the monopole is proportional to the area of the D2-brane, $m \propto \frac{L_6 \phi}{g_s} \sim \phi/g^2$ in agreement with semiclassical field theory computations.

Including D0-brane instantons stretched between the two NS5-branes and coincident with the color branes [23, 21, 37] allows us to compute instanton effects in the gauge theory. To see why, note that since the RR 1-form A_{RR} couples to the D4-brane worldvolume fields via

$$\mathcal{L} \supset A_{RR} \wedge F \wedge F, \quad (2.10)$$

the worldline of an instanton configuration of the gauge fields (which is extended in x^6) will inherit an electric coupling to A_{RR} proportional to its instanton charge, precisely the defining feature of a D0-brane. Further, it is also clear from this coupling that the theta

angles of the gauge theory are associated with the Wilson line of A_{RR} along x^6 [38].

Since the D-instantons are BPS, they locally break half the $\mathcal{N} = 2$ supersymmetry. The broken supersymmetries can no longer ensure force cancellation between the charged ends of the D4-branes coincident with them. Thus, two color D4-branes approaching each other along the Coulomb branch of pure $SU(N_c)$ SYM will find that D0 instanton corrections give rise to repulsive forces between them that prevent them from coinciding. Note that this effect becomes important only when the branes approach each other, as the $U(1)$ gauge theory associated with a single D4-brane cannot accomodate instantons! Therefore the full gauge symmetry of the theory is never restored on the Coulomb branch. In addition, the presence of the D0-brane instanton affects the D2-brane monopole in the region between the approaching color branes giving rise to non-perturbative corrections to its mass of order Λ . These facts [37] provide physical intuition for the phenomena encountered in the exact solution of [18].

These considerations are modified if we add quarks, as color branes attached to flavor branes (from the perspective of the NS5-branes, this looks like opposite charges annihilating) no longer support finite action instantons on their worldvolumes. Thus, there is no obstruction to such connected branes becoming coincident, and one expects loci in the moduli space with enhanced gauge symmetry. As we will review in section 2.2, such loci are indeed present and are the focus of our work.

2.1.6 Quantum gauge theory and M-Theory

By lifting the above brane configuration to M-theory [22], we can compute exactly some of the non-perturbative (in g_s) quantum corrections discussed above, and thereby gain considerable insight into the origin of quantum phenomena in SQCD. The D0-brane related corrections we have argued for above will have a much more natural interpretation in terms of the physics of the extra dimension. In particular, we consider the above IIA brane configuration in the limit of large g_s and L_6 with $\frac{1}{g^2} = \frac{L_6}{R}$ fixed, where the classical description of M-theory is valid.

In M-theory, the D4-brane is an M5-brane which wraps the M-theory circle ($x^{10} \sim x^{10} + 2\pi$) once while an NS5-brane is an M5 brane which does not wrap the x^{10} direction. The collection of D4 and NS5-branes can therefore be described as a *single M5-brane* $\mathbb{R}^{3,1} \times \Sigma$ which fills the x^0, x^1, x^2, x^3 space and is a Riemann surface Σ in the x^4, x^5, x^6 , and x^{10} directions.

In fact, if we introduce the new holomorphic coordinate,

$$t = \exp \left(\frac{-x^6}{R} - ix^{10} \right) \quad (2.11)$$

where R is the radius of the M-theory circle, we can construct Σ as the vanishing locus of a polynomial $F(v, t)$ [22]. As each NS5-brane corresponds to an asymptotic branch $v \rightarrow \infty$ of the surface regarded as a multivalued function $t(v)$, we expect that $F(v, t)$ is a quadratic polynomial in t . Flavor branes are seminfinite D4-branes at some $v = m_i$ and therefore correspond to asymptotic regions in the surface where $v \rightarrow m_i$ and $t \rightarrow 0$. Color branes (generically) are handles connecting the two NS5-branes which also wrap the M-theory circle. Regarding the surface as a multivalued function $v(t)$ for small t , we expect it to have N_c roots, implying that $F(v, t)$ should be a polynomial of degree N_c in v . Ignoring the M-theory circle, $|t|$ is a double-valued function of v and we expect that the two branches should meet at the semiclassical positions of the color branes $v = \phi^a$. The above considerations fix Σ up to a constant Λ , which can be identified with the dynamically generated QCD scale of the theory (see [22]),

$$t^2 + t \prod_{a=1}^{N_c} (v - \phi^a) + \Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (v - m_i) = 0. \quad (2.12)$$

Note that Σ is precisely the Seiberg-Witten curve [18] of the gauge theory! For example, the lift of the brane configuration shown in Fig. 2.3 is the single M5-brane pictured in Fig. 2.4.

The primary benefit of the lift to M-theory is that its classical geometry contains the Seiberg Witten curve and therefore encodes all the instanton effects in the gauge theory. For instance, the running of the gauge coupling at a generic point on the Coulomb branch is lifted in M-theory to the running of the full complex coupling including the effective theta

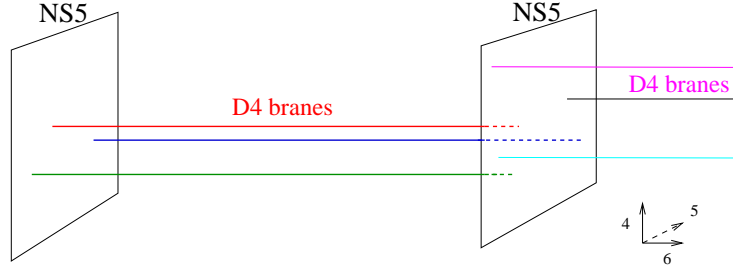


Figure 2.3: IIA realization of the Coulomb branch of $\mathcal{N} = 2$ $SU(3)$ SQCD with 3 flavors

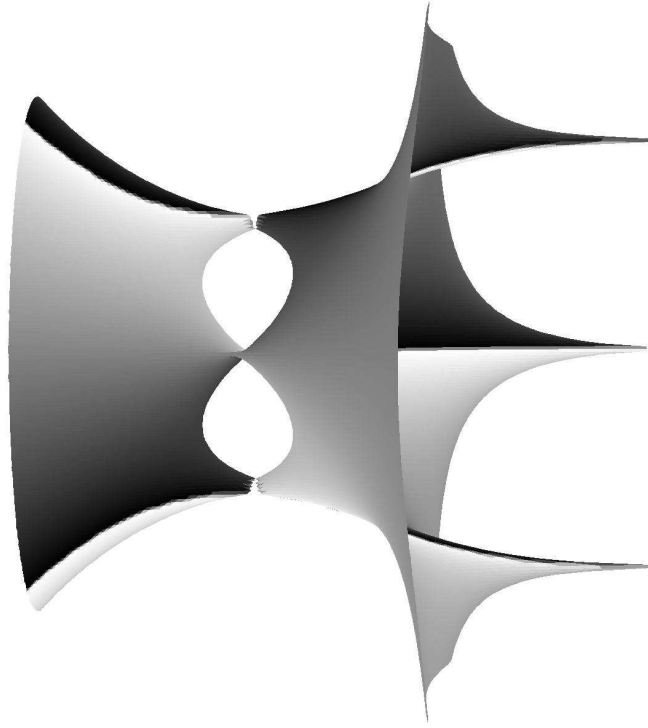


Figure 2.4: The M-theory lift of the above IIA configuration to a single M5-brane. The directions x^4 , x^5 , and x^6 are shown explicitly while the x^{10} coordinate is parameterized by darkness.

angle,

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \sim \log v \quad (2.13)$$

Further, the thickening of the D4-branes in v is a manifestation of the instanton induced repulsion between color brane ends as well as the corresponding corrections to the masses of BPS states (which are associated with BPS open M2-branes ending on the M5-brane). We can see that they generically meet over an interval of order Λ in v , reflecting the observation that the adjoint vevs are not good coordinates everywhere in moduli space.

Also, the $U(1)_R$ anomaly is particularly easy to see in the M-theory picture. Recall that D4-branes wrap the M-theory direction and end on the NS5-branes, which do not wrap the M-theory direction. This means that the end of a D4-brane on an NS5-brane is a vortex in the embedding coordinate of the NS5-brane in the M-theory direction (see Figs. 2.4 and 2.7), as we had mentioned earlier. In particular, at large v if one follows a circle along the NS5-brane whose interior contains all of the D4-branes, this circle will wrap the M-theory direction as many times as there are colors minus flavors attached to this NS5-brane. Thus the naive $U(1)_R$ rotational R-symmetry of the brane must be combined with a simultaneous rotation of the M-theory circle. Such a redefinition is not possible with both NS5-branes as the x^{10} redefinitions would have to be in opposite directions for the two branes and so the $U(1)_R$ is anomalous. There is a residual $\mathbb{Z}_{2N_c - N_f}$ symmetry in the $U(1)$ redefined to include a rotation of the M-theory circle in opposite directions on both NS5-branes. For these special angles the two redefinitions only disagree by a multiple of 2π in the M-theory direction. The reader may verify these claims visually by considering the asymptotic x^{10} dependence of the regions corresponding to the NS5-branes (large positive and negative values of x^6) on the M5-branes pictured in Figs. 2.4 and 2.7.

Another benefit of the M-theory lift is that all the matter in the gauge theory is realized in M-theory by open M2-branes ending on the M5-brane. In reducing to IIA, the M2-branes which wrap the M-theory circle carry fundamental string charge, while unwrapped M2-branes are D2-branes. The M2-brane configurations corresponding to BPS states were analyzed critically in [40, 39]. As predicted in [22], they found that BPS states were M2-

branes with minimal areas in their homology class, as we will review in greater detail in section 2.4.

For example, baryons are M2-branes [49] with $k + 1$ boundaries as seen in Figure 2.5. One boundary wraps k color branes and the other k boundaries wrap flavor branes. In the IIA limit, the baryon is a collection of k quarks attached by a k -string [23, 51]. Mesons are tubular M2 branes which start by wrapping one flavor brane, connect to a color brane, and then extend in the space directions (x^1 , x^2 , and x^3) before wrapping another flavor brane.

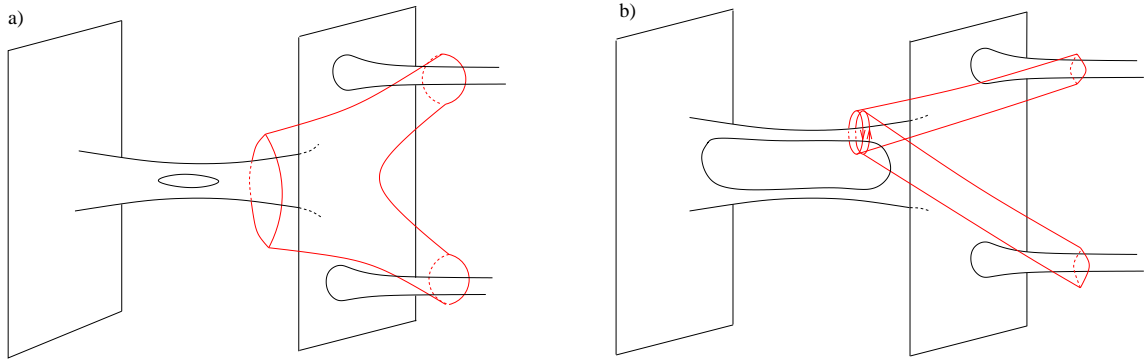


Figure 2.5: a) a baryon which wraps 2 color and 2 flavor branes in $SU(2)$ SQCD with 2 flavors (b) a meson which wraps 2 flavor branes and wraps and unwraps one color brane

We certainly have gained a great deal in this lift, but we don't quite get it for free. Since this lift involves going to large string coupling, one cannot be sure that the low energy effective theory on the M5-brane is the gauge theory we started with. In fact, it is actually a six dimensional theory. To see this, note that the M theory limit is exactly the opposite limit needed to obtain the 4D gauge theory. It requires taking the radius of the M-theory circle $R = g_s l_s$ as well as L_6 large leaving $\frac{1}{g^2} = \frac{L_6}{R}$ fixed. However, since unbroken supersymmetries protect certain holomorphic quantities (like the masses of BPS states and superpotentials) from perturbative g_s quantum corrections, one can still use classical computations involving the M5-brane to determine them exactly. The breakdown of the M-theory picture for the computation of non-holomorphic quantities (like the masses of non-BPS states) can then be understood as due to the presence of KK modes (dynamical

D0-branes and A_6 fluctuations) which become light and strongly coupled in this limit [23].

For example, one can solve the $\mathcal{N} = 2$ gauge theories constructed above at generic points in their moduli spaces as their effective actions are governed by holomorphic quantities ($\mathcal{N} = 2$ prepotentials \mathcal{F}) which can be computed exactly using Σ [22]. Upon breaking to $\mathcal{N} = 1$, the Kähler potential is no longer protected from such corrections, and we can no longer expect such dramatic results from M-Theory. However, as the superpotential of the $\mathcal{N} = 1$ theory is still protected, we will see that we can still learn what we need about $\mathcal{N} = 1$ SQCD from M-theory.

2.1.7 Soft breaking to $\mathcal{N} = 1$ SQCD

In order to explore non-trivial physics associated with confinement, flavor-symmetry breaking, and Seiberg Duality, we must, of course, break the supersymmetry down to $\mathcal{N} = 1$. We do this by rotating one of the NS5-branes,

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \quad (2.14)$$

This results in a brane configuration which preserves only four supercharges [41] and softly breaks $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. We will refer to the rotated NS5 brane as the NS5 $_{\theta}$ brane. The NS5 $_{\pi/2}$ brane is commonly referred to as the NS5' brane in the literature.

The symmetry breaking classical superpotential generated by this process can be understood as follows. After the rotation, color branes at generic positions in v no longer minimize their lengths. In fact, to reach equilibrium, all color branes must either slide to $v \sim O(\Lambda)$ or attach to flavor branes. Since translations along v (which correspond to adjoint scalar vevs) now cause the color branes to stretch, the $\mathcal{N} = 1$ chiral multiplets containing these adjoint scalars acquire a mass μ via the superpotential term

$$W \supset \mu \operatorname{Tr} \Phi^2 \sim \tan \theta. \quad (2.15)$$

If r color branes connect to flavor branes, classically the only surviving vacuum is

$$\phi^i = m_i \text{ for } i = 1, \dots, r \quad \phi^a = 0 \text{ for } a = r + 1, \dots, N_c. \quad (2.16)$$

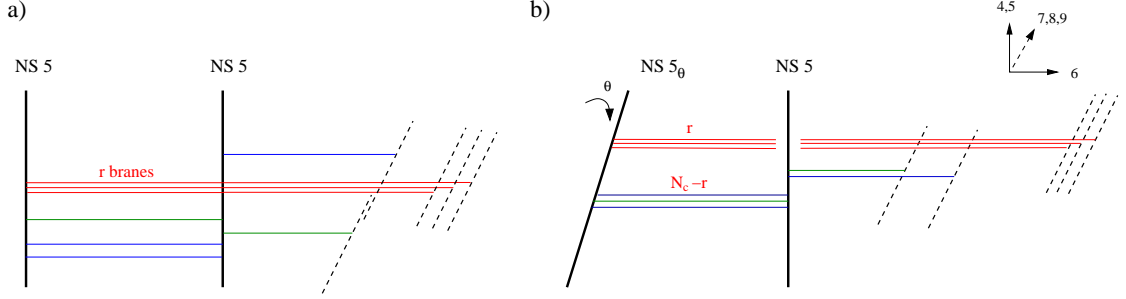


Figure 2.6: a) r color branes and flavor brane connects. b) After rotating the five brane, color branes which are not connected to flavor branes move to the origin of v . This is the realization of the r vacua in [24].

However, when quantum corrections are considered the surviving vacua are those where $N_c - r$ or $N_c - r - 1$ monopoles and dyons become massless and condense with vevs of order $O(\mu\Lambda)$. In the M theory picture, this occurs when the bounding cycles of the corresponding M2-branes degenerate, as in Fig. 2.7.

More precisely, the vacua which survive the $\mu\text{Tr}\Phi^2$ perturbation are those in which the cycles that have degenerated result in an M5-brane configuration $\Sigma \times \mathbb{R}^4$ such that Σ is of genus zero, as argued in Refs. [18, 23, 53]. We will outline here the argument for pure $SU(N_c)$ gauge theory, though the argument in Ref. [53] extends to the general case.

A genus zero curve is birationally equivalent to \mathbb{CP}^1 , which means that one can find a rational parameterization of such a curve in terms of an auxiliary complex parameter λ . We construct the parameterization by considering the brane setup pictured in Fig. 2.7. The portions of the M5-brane corresponding to the two NS5-branes are characterized by the fact that they are the only asymptotic regions corresponding to $v \rightarrow \infty$ and $|\log t| \rightarrow \infty$. Thus, without loss of generality, we can take $v = \lambda + c\lambda^{-1}$ for some complex constant c . The two NS5-branes are then just the regions $\lambda \rightarrow \infty$ and $\lambda \rightarrow 0$. Further, as the M5-brane must wrap the M theory direction N_c times in opposite directions as we make a loop around each NS5-brane, we expect that $t \sim \lambda^{N_c}$ and that $c^{N_c} = 1$. In order to rotate the NS5-brane corresponding to $\lambda \rightarrow 0$, we pick $w = \mu\lambda^{-1}$ for some complex number μ . With this choice,

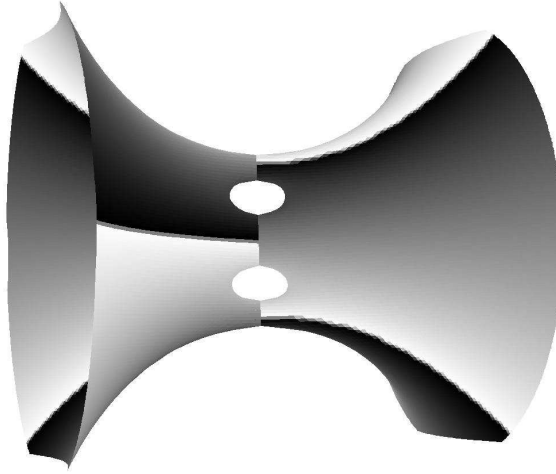


Figure 2.7: An M5-brane configuration corresponding to $\mathcal{N} = 2$ $SU(3)$ SYM near a vacuum which survives upon softly breaking to $\mathcal{N} = 1$. The two small holes are the degenerating cycles. The M2-branes corresponding to nearly massless monopoles are disks bounded by these cycles. The M-theory direction is parameterized by darkness.

we see that in the limit $\lambda \rightarrow 0$, $v \approx \mu w$, which implies that at high energies (large v and w) this configuration indeed reduces to a rotated NS5-brane in IIA. Since there are N_c such curves corresponding to the choice of c above, we do in fact find all the vacua this way (as per Witten Index computations, which show $\text{Tr}(-1)^F = N_c$).

To preserve SUSY the flavor branes must continue to extend along the x^0 , x^1 , x^2 , x^3 , and x^6 directions, in particular they cannot rotate into the w directions. Therefore when the NS5 $_\theta$ brane rotates, the flavor branes ending on it must translate in w , sliding along the corresponding D6-brane. This translation corresponds to meson vevs [50, 53].

2.2 Summary of Field Theory Results

The dynamics of the $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ gauge theories constructed above and the dynamical breaking of flavor symmetry have been studied in detail in Refs. [42, 43,

54]. Here we briefly summarize the results.

The theory with N_f massless quark hypermultiplets has $U(N_f)$ flavor symmetry, $SU(2)_R$ symmetry, and a non-anomalous discrete $\mathbb{Z}_{2N_c-N_f}$ subgroup of $U(1)_R$. We are interested in the $\mathcal{N} = 1$ perturbation of the theory by the adjoint mass term $\mu \text{Tr } \Phi^2$. The moduli space contracts to the set of points that give maximally degenerate (genus zero) Seiberg–Witten curves.

In the semi-classical regime with a large adjoint VEV, there are ‘t Hooft–Polyakov magnetic monopoles. Zero modes of the quarks around a monopole generate flavor quantum numbers for the magnetic monopoles. It was shown that they come in completely anti-symmetric rank- r tensor representations with $_{N_f}C_r$ multiplicities.

The strongly coupled regime was studied with a variety of techniques. When $N_f < N_c$, there are vacua parameterized by an integer $r = 0, 1, \dots, [N_f/2]$ ($[x]$ is the Gauss’ symbol) where the flavor symmetry is broken dynamically as $U(N_f) \rightarrow U(r) \times U(N_f - r)$. If $r < N_f/2$, the physics around the vacuum can be described by an IR free effective Lagrangian [24] (a “magnetic dual” to the asymptotically free semiclassical $SU(N_c)$ description) with $SU(r) \times U(1)^{N_c-r-1}$ gauge group. N_f “magnetic quark” hypermultiplets transform as the fundamental representation of $SU(r)$ while there are “magnetic monopoles” for each of the “magnetic” $U(1)$ factors. When perturbed by the adjoint mass term, all gauge groups are Higgsed by the condensates of magnetic objects, corresponding to the confinement of the electric theory. It was argued in Ref. [42, 43] that the semi-classical monopoles in the rank- r anti-symmetric tensor representation smoothly match to the baryonic composites of magnetic quarks of the low-energy $SU(r)$ theory based on circumstantial evidence. Upon mass perturbations, one can count the number of vacua:

$$\mathcal{N}_1 = (2N_c - N_f)2^{N_f-1}, \quad (2.17)$$

originating from r -vacua ($r \leq [N_f/2]$) with $(2N_c - N_f)$ copies due to the $\mathbb{Z}_{2N_c-N_f}$ symmetry. Therefore *the flavor symmetry breaking and confinement have a common origin in these theories*: condensation of magnetic objects with non-trivial flavor quantum numbers. Strictly speaking, however, the existence of monopoles in the anti-symmetric tensor rep-

representations was demonstrated only in the semi-classical regime and its extrapolation to the strongly coupled regime and the matching to the baryonic composite was a conjecture. When $r = N_f/2$ (possible obviously only when N_f is even), the low-energy magnetic gauge group is superconformal with an infinitely strong coupling $\tau = -1$. Due to some reason, the same low-energy effective action seems to describe the dynamics of flavor symmetry breaking even though there is no weakly coupled description of the theory.

When $N_f > N_c$, there is a new vacuum without flavor symmetry breaking. It is at the same point on the moduli space as the $r = N_f - N_c$ vacuum, while the finite quark mass perturbation shows that there are additional

$$\mathcal{N}_2 = \sum_{r=0}^{N_f-N_c-1} (N_f - N_c - r)_{N_f} C_r \quad (2.18)$$

vacua with unbroken flavor symmetry.

2.3 Seiberg Duality from M-Theory

As we noted in the last section, the effective theories at the r -vacua which survive SUSY breaking are actually IR free theories for $r < N_f/2$. In particular, at the baryonic root we find a weakly coupled theory whose low energy physics is well described by an IR free $SU(\tilde{N}_c)$ gauge theory, where $\tilde{N}_c = N_f - N_c$. This is very reminiscent of Seiberg duality in $\mathcal{N} = 1$ theories [17]. In [24], this observation was used to “derive” $\mathcal{N} = 1$ Seiberg duality by mass perturbing the $\mathcal{N} = 2$ theory³. We will consider this scenario using M-theory.

Seiberg duality was first realized in IIA string theory in Ref. [48] via the exchange of two NS5-branes, as illustrated in Fig. 2.8. To avoid a singular configuration the authors first displaced one of the NS5-branes in the 7 direction, which corresponds to turning on a Fayet-Illipolous term in the field theory description. This process corresponds to Seiberg duality with one caveat, the full $U(N_f) \times U(N_f)$ flavor symmetry is not realized in the

³Unfortunately, the result was not quite Seiberg duality as there was an extra non-renormalizable coupling in the effective action associated with integrating out the massive adjoint scalar which becomes relevant in the low energy limit.

above IIA configuration nor in its M-theory lift.

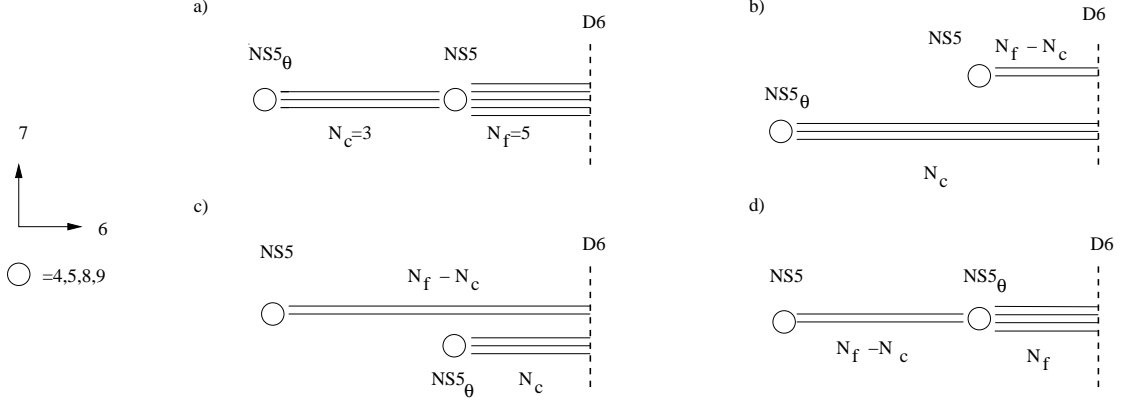


Figure 2.8: In IIA Seiberg duality is realized by brane exchange.

An alternate proposal for realizing Seiberg duality via string/M theory was made later that year by Schmaltz and Sundrum [55]. Using the M-theory lift of the above brane setup, they found that Seiberg duality could be understood by taking the $\Lambda \rightarrow 0$ and $\Lambda \rightarrow \infty$ limits of the resulting single M5 brane configuration. In particular, they found that the two limits were related by an exchange of branes and therefore correspond to an electric theory and its magnetic dual. However, their arguments depended on turning on finite bare quark masses.

The following year this scenario was clarified and extended to the massless case by Hori [52]. He observed that in M-theory, the M5-branes can be crossed with no singularity even without the Fayet-Illiopolous parameter. In fact, at the root of the baryonic branch, when $N_f \geq N_c + 1$ the M5-brane consists of two connected components [53] and the duality corresponds to simply translating one past the other along x^6 . The x^6 coordinate of a connected component does not affect the field theory description and so Hori argues that this process clearly preserves its universality class.

Most recently, in Ref. [46] Seiberg duality was understood within the context of geomet-

ric engineering as a birational flop in a T dual description consisting of D5-branes which are dual to the D4 branes in IIA and degenerations in the T-dualized circle which are dual to the NS5-branes.

We propose a new description of Seiberg duality which has no moving branes and is valid at any fixed, finite value of Λ . Consider the case $N_f > N_c$ at the root of the baryonic branch in an $\mathcal{N} = 2$ theory with no bare quark masses. Then according to Ref. [53] the connected components of the M5-brane are described by

$$t = v^{N_c} \quad \text{and} \quad t = v^{\tilde{N}_c}, \quad \tilde{N}_c = N_f - N_c \quad (2.19)$$

as seen in Figs. 2.9 and 2.10. These two M5 branes intersect at $N_c - \tilde{N}_c$ points.

The crucial realization is that the reduction of this configuration to IIA and in particular the x^6 location of the NS5-branes is not uniquely defined [22]. Rather, we claim, that the effective location of the NS5-branes depends on the energy scale probed, E . We consider charged hypermultiplet matter corresponding to M2-branes of disk topology stretched between the two branches of the M5-brane. It is clear from Fig. 2.10 that the area of such an M2-brane (and therefore its energy) is proportional to the distance in v between the two branches of M5-brane at its position in x^6 . This restricts the regions on the M5-brane that a quark of a given energy can probe. Thus, for such a probe, one may effectively place an NS5-brane at the intersection of the corresponding M5-brane and $v \sim E$. A high energy $E_1 \gg \Lambda$ (semiclassical) M2-brane probe is restricted to probe large features at small x^6 . Thus, to such a probe, the configuration consists of an NS5-brane with an NS5 $_\theta$ brane on its left, N_c color branes connecting them and N_f semi-infinite flavor branes which extend to the right. That is, the reduction to type IIA is Fig. 2.8a, which can roughly be obtained by drawing an NS5-brane wherever the line E_1 crosses an M5 brane. This corresponds to the $SU(N_c)$ asymptotically free electric theory. However, low energy quarks can only exist at sufficiently large x^6 . Thus, a low energy $E_2 \ll \Lambda$ M2 brane probe corresponding to a charged quark is only sensitive to the configuration of the M5-brane at large x^6 , the right side of Fig. 2.9. Thus, if we consider the portion of the M5-brane configuration accessible to such a probe, we would find that the corresponding reduction to IIA at E_2 is Fig. 2.8d.

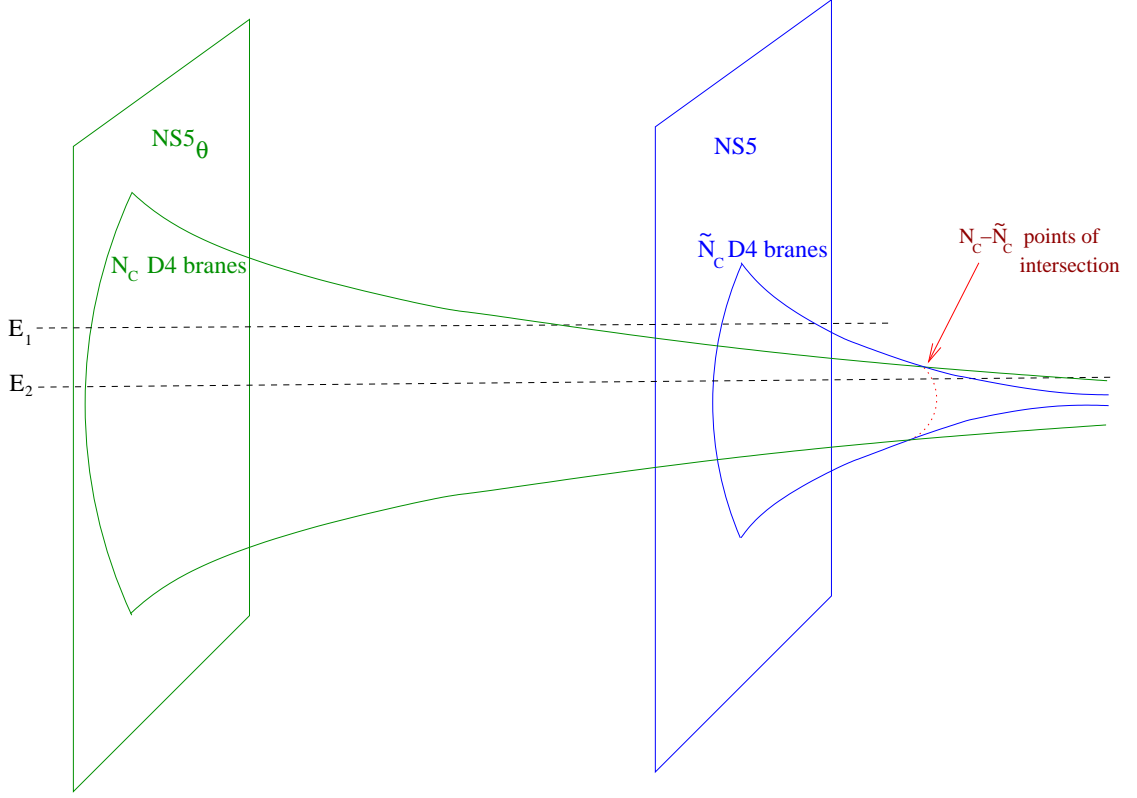
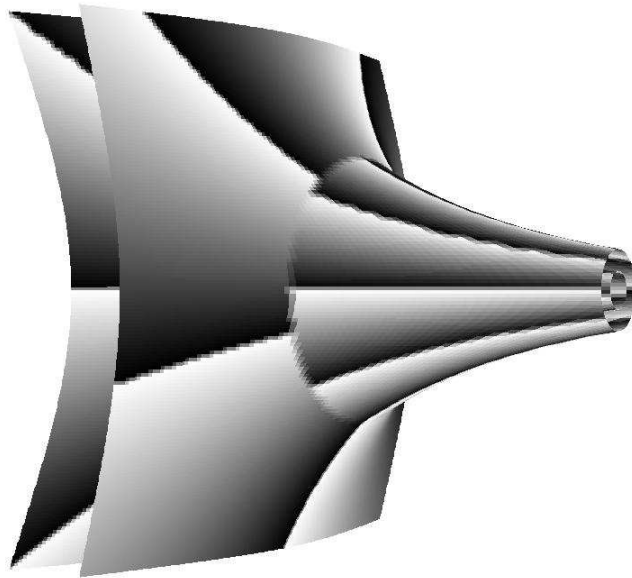


Figure 2.9: At the root of the baryonic branch, the low energy physics (to the right of the intersection) is the $SU(\tilde{N}_c) \times U(1)^{N_c - \tilde{N}_c}$ magnetic theory.

Now, note that the two M5-branes cross at the QCD scale Λ , and at energy scales below this, that is, further right in the figure, the M5-brane is to the left of the $M5_\theta$ brane. Thus a probe at energies below the QCD scale will see the two M5 branes interchanged, which is the usual description of the magnetic theory. This theory is IR free because the branes separate as v increases, and has a Landau pole where the branes cross.

Figure 2.10: Baryonic root of $SU(8)$ with 13 flavors

2.4 Flavor Symmetry Breaking

2.4.1 Flavored magnetic monopoles

Recall that a flavorless magnetic monopole (or dyon) in a IIA realization of the electric picture is a D2-brane with disk topology bounded by a circle which extends along one color brane, down an NS5 brane, back along another color brane and then finally back along the other NS5-brane, as in Fig. 2.1. More generally magnetic monopoles may be charged under the $U(N_f)$ flavor symmetry. In this case the monopole may include fundamental strings extending from the D2-brane to a flavor brane. Another version of the s-rule states that at most one such fundamental string can extend from a given monopole to a given flavor brane, which identifies these strings as excitations of the fermionic zero-modes present in flavored monopoles.

The M-theory lift of this monopole configuration is a single M2-brane with the topology of a disk. Thus, its boundary is a circle which wraps the M-theory direction a total of zero times, as seen in Fig. 2.11.

The microscopic mechanism behind the flavor symmetry breaking of Refs. [42, 43, 54] is the condensation of magnetic monopoles in an antisymmetric tensor flavor representation drawn in Fig. 2.11a. The transformation properties under $U(N_f)$ can be read off from the brane cartoon. If the monopole does not wrap any flavor branes it transforms as a flavor singlet. A monopole that wraps one of the flavor branes transforms in the fundamental representation, while wrappings of more than one flavor brane transform in the antisymmetric representation of the flavor group. To see why this representation is antisymmetric, notice that in IIA, if the D6-branes are moved between the NS5-branes, a monopole consists of a D2-brane connected by strings to D6-branes. The s-rule provides an exclusion principle, restricting the number of strings connecting a monopole to a D6-brane to 0 or 1. As a consistency check on this picture, notice that there are 2^{N_f} configurations of wrappings which agrees with the known number of states in the representation.

In order to understand how semiclassical magnetic monopoles in the UV theory are related to the IR degrees of freedom, we can consider an M2 brane corresponding to a high energy monopole configuration and follow its decay into the IR. Deforming the UV theory away from the baryonic root to clearly visualize its charges, such a monopole is pictured in Fig. 2.11a. We deform back to the baryonic root and then allow it to decay, requiring that its wrappings (i.e. charges) are preserved. Its energy will become sufficiently low that it is best described using the dual magnetic description, that is, with the NS5 branes switched. Now, if we deform the IR theory to a generic point in its Coulomb branch and keep track of the wrappings, this configuration corresponds to magnetic baryons as seen in Fig. 2.11b. The magnetic theory is IR free, and so these baryons decay into magnetic quarks whose condensation provides the order parameter of the flavor symmetry breaking.

It is also easy to see that the correlation between the electric charge and chirality of dyons discussed for $SU(2)$ gauge theories in Ref.[19] follows from the fact that the monopole is topologically a disk. All monopoles with even chirality come from monopoles whose

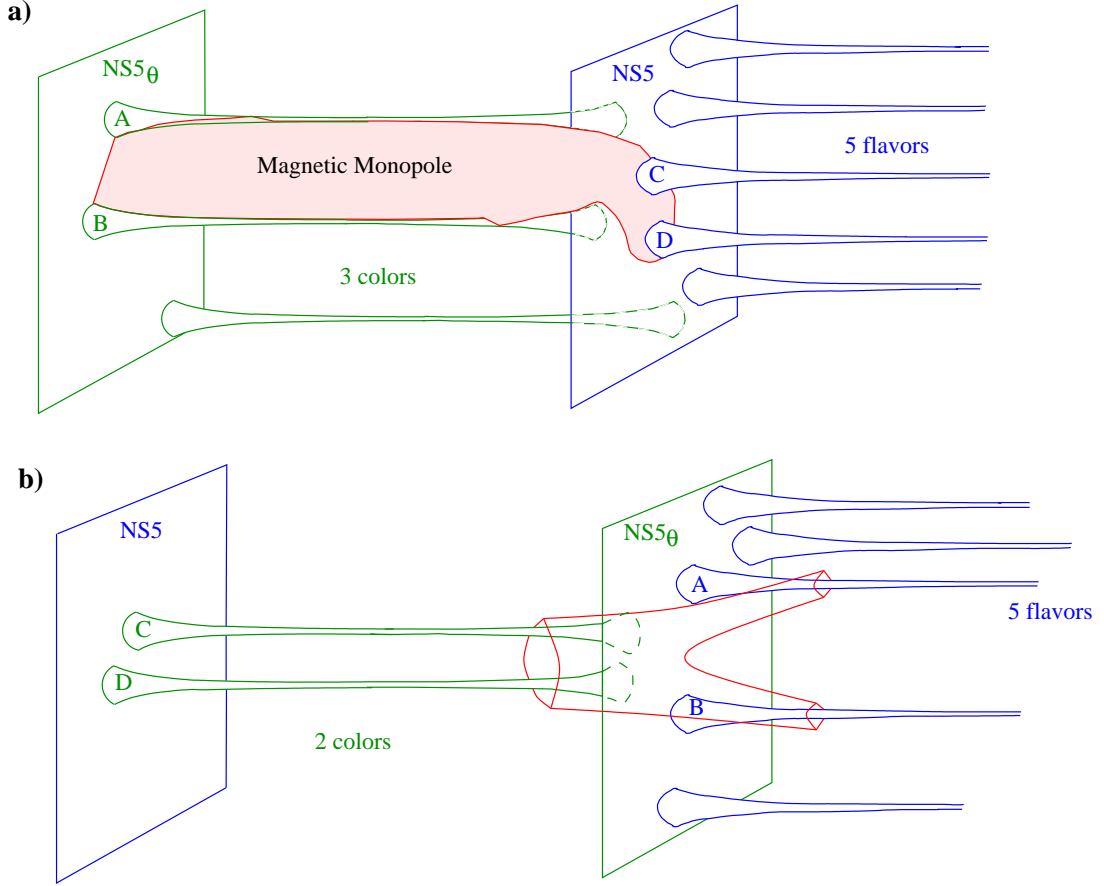


Figure 2.11: a) A magnetic monopole in an antisymmetric tensor flavor representation. b) In the dual magnetic picture it becomes a baryon.

boundary wraps an even number of flavor branes and therefore the M-theory direction an even number of times along the M5 brane as well. Each flavor brane wrapping introduces a hypermultiplet into the antisymmetric representation and hence the monopole has chirality $(-1)^H = 1$. Each color brane wrapped yields a unit (using the conventions of [19]) of electric charge and so the monopole acquires an even electric charge. This argument works similarly for odd chirality and odd electric charge.

2.4.2 Symmetry breaking pattern

As we reviewed in section 2.2, field theory calculations in a variety of limits showed that flavor symmetry is generically broken

$$U(N_f) \rightarrow U(r) \times U(N_f - r) \quad (2.20)$$

in softly broken $\mathcal{N} = 1$ asymptotically free ($N_f < 2N_c$) SQCD in the limit that bare quark masses vanish. We will draw string and M-theory realizations of these limits and use them to reproduce the qualitative results of several field theory calculations. In particular, in the semiclassical limit, we will relate r to the number of color branes attached to flavor branes.

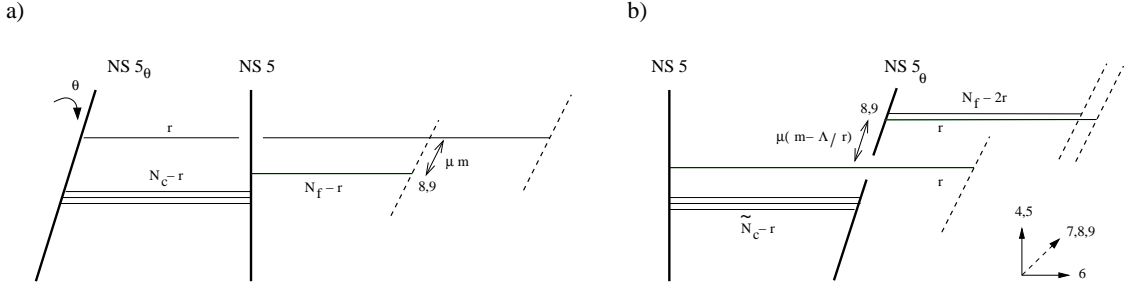


Figure 2.12: (a) Semiclassically when SUSY is broken by rotating the NS 5_θ brane, r color branes connect to flavor branes while the rest slide to $v = 0$ (b) The dual magnetic description of the nonbaryonic branches can be understood similarly to the semiclassical case.

2.4.3 Semiclassical analysis

Following Refs. [42, 43, 54] we begin by considering bare quark masses much larger than the QCD scale so that a semiclassical (IIA) analysis is valid. This will allow us to count the total number of vacua for comparison with later calculations. Recall from section 2.1.7 that after rotating the NS 5_θ brane all color branes must either slide to $v = 0$ or connect to a flavor brane as in Fig. 2.12a. The number of color branes connecting to flavor branes, r , clearly can neither exceed the number of color branes nor the number of flavor branes. We

illustrate the simple case of the vacua arising this way with the bare quark masses all equal $m = m_i \gg \Lambda$ in Appendix A for the case of $SU(3)$ with $N_f = 4$.

To count the number of vacua with generic quark masses, notice that r flavor branes can attach to color branes in $\binom{N_f}{r}$ ways (recall that there is no combinatoric factor from choosing which color branes to attach as these choices are gauge equivalent), leaving $N_c - r$ color branes which form a line⁴ [51] centered at $v = 0$, as shown in Fig. 2.7. This line can be rotated by integer multiples of $\pi/(N_c - r)$ without affecting the x^{10} coordinates asymptotically far away, corresponding to the anomaly-free R-symmetry subgroup. The $N_c - r$ inequivalent orientations of the line result in $N_c - r$ different vacua, in agreement with the Witten index of this theory. Thus the total number of semiclassical vacua (assuming all bare quark masses are distinct and nonvanishing) is

$$\mathcal{N}_{sc} = \sum_{r=0}^{\min(N_c+1, N_f)} (N_c - r) \binom{N_f}{r} \quad (2.21)$$

in agreement with computations from field theory considerations in Refs. [42, 43, 54].

2.4.4 Nonbaryonic branches

Semiclassically flavor symmetry is broken by meson vevs equal to $\mu m_i \sim \mu \tan \theta$ which is the distance in the w plane shown in Fig. 2.12a. These vevs vanish when the bare quark masses vanish, apparently restoring the explicitly broken flavor symmetry. However we will see that, if we include quantum effects, in some vacua the flavor symmetry remains broken even in this limit. For simplicity let all of the bare quark masses be equal $m = m_i \ll \Lambda$.

The $r = N_f/2$ theory is superconformal in the IR and generally difficult to understand. An example of M5-brane associated with such a vacuum is drawn in Fig. 2.13. The two branches, corresponding to different NS5-branes upon reduction to IIA, intersect at exactly two points. Cross-sections to the left and right of these two singularities are seen in the first and second lines of Fig. 2.14. They are topologically distinct and correspond to distinct reductions to IIA, yielding different effective field theories.

⁴Actually they form an ellipse whose semi-minor axis scales with θ . This ellipse degenerates to a line

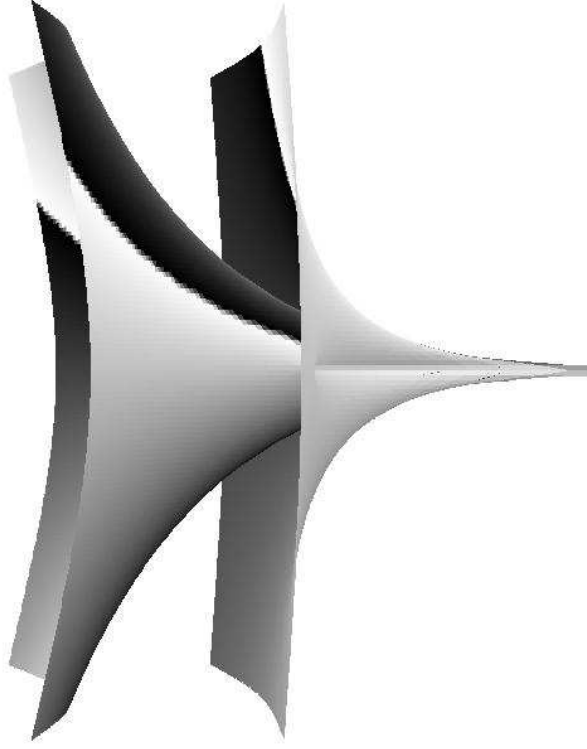


Figure 2.13: This M5-brane configuration corresponds to the $r = 2$ nonbaryonic root of $SU(3)$ with 4 flavors. In the IR (the right side of the picture) a reduction to IIA produces two parallel, almost coincident NS5-branes indicating that the theory is strongly coupled. The fact that the distance between the branes converges indicates that the IR theory is superconformal.

We will be interested in the IR free case of $0 < r < N_f/2$, corresponding to nonbaryonic branches. The fundamental degrees of freedom must be magnetic because in the UV this theory has a Landau pole (M5-branes cross) separating it from the semiclassical region with electric degrees of freedom. Thus flavor symmetry breaking can only be caused by magnetic quark vevs, which like meson vevs (meson vevs are quadratic in quark vevs) correspond to distances in the w plane. Semiclassically this distance was μm_i and so vanished when the

when $\theta = 0$ and a circle with an A_{N_c-r-1} singularity in its center at $\theta = \pi/2$.

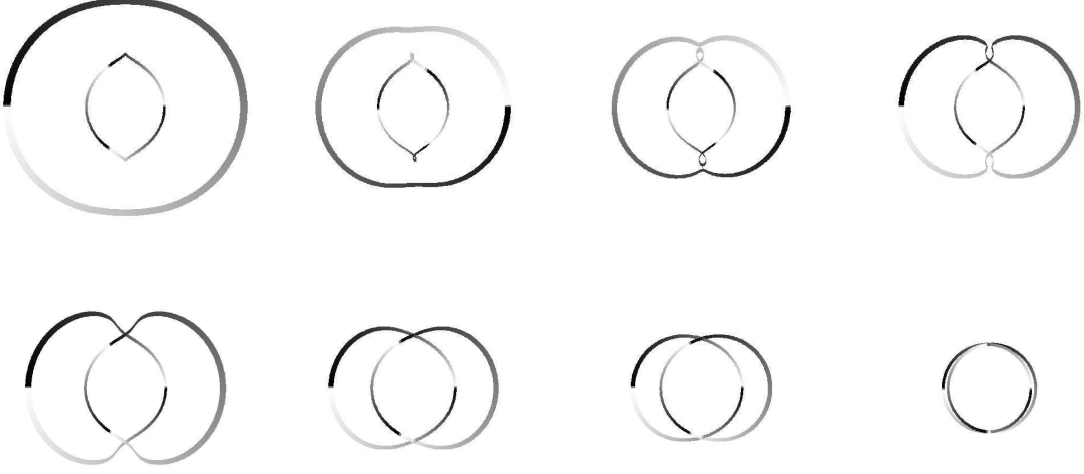


Figure 2.14: Above are M5-brane cross-sections at constant x^6 corresponding roughly to constant energy scales of the $r = 2$ nonbaryonic root of the $SU(3)$ theory with 4 flavors and no bare quark masses. Reading from left to right and top to bottom the energy scale of the corresponding M2-brane probes decreases. The M theory direction is parameterized by darkness.

bare quark masses were taken to zero. Quantum mechanically these flavor branes have width $O(\Lambda)$, critically changing the distances between them. As a result the r nonvanishing magnetic quark vevs

$$q = \sqrt{\mu(m_i - \Lambda/r)} \quad (2.22)$$

do not vanish when the bare masses are eliminated, as seen in Fig. 2.12b. These quark vevs break the flavor symmetry

$$U(N_f) \rightarrow U(r) \times U(N_f - r). \quad (2.23)$$

These vevs also break the global $\mathbb{Z}_{2N_c - N_f}$ symmetry and so there must be $2N_c - N_f$ copies of this configuration. When $r = 0$, $q = \sqrt{\mu m_i}$ and flavor symmetry is unbroken. Again there are $\binom{N_f}{r}$ ways to choose r quarks, leading to a total of

$$\mathcal{N}_1 = (N_c - \tilde{N}_c) 2^{N_f - 1} \quad (2.24)$$

vacua of this type. Notice that when $N_f < N_c$, \mathcal{N}_1 agrees with \mathcal{N}_{sc} and therefore by supersymmetry this is a complete classification of the vacua.

2.4.5 Baryonic branch

The above analysis is incomplete when $N_f \geq N_c$ because every color brane can be broken by a D6-brane and the two halves displaced from each other along the w plane by a distance (baryon vev) exactly canceling the displacement measured by the dual quark vev. Clearly this requires $N_f \geq N_c$ because N_f is the number of D6-branes while N_c is the number of color branes, and each color brane requires a D6-brane along which to break, as illustrated in Fig. 2.15.

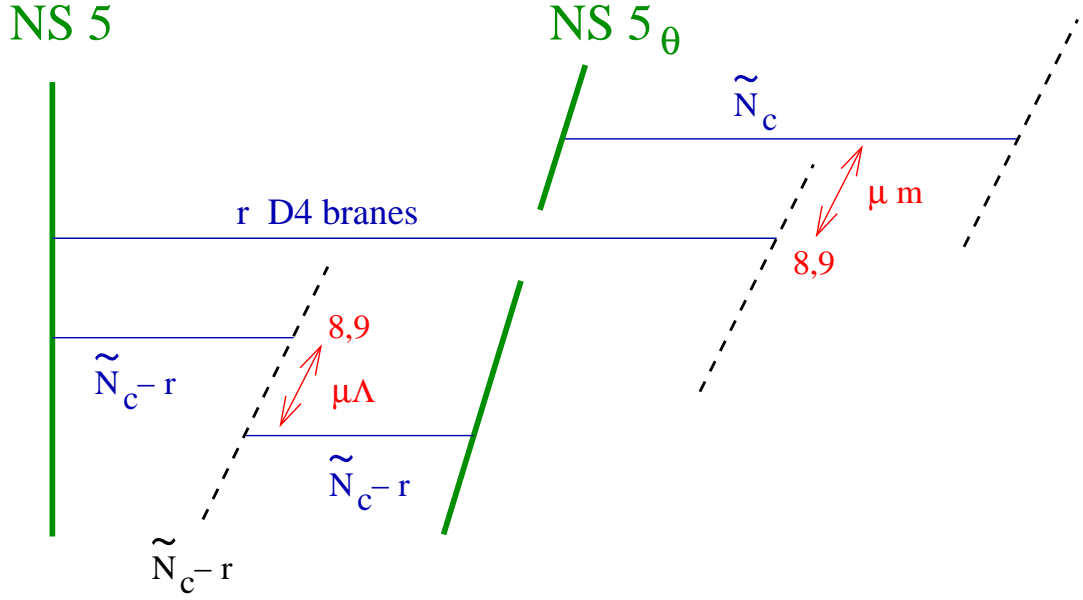


Figure 2.15: On the baryonic branch, the baryon vev $\mu\Lambda$ is realized as a relative slide of the two halves of the configuration along a collection of D6-branes.

More concretely, at the root of the baryonic branch the unattached $N_c - \tilde{N}_c$ color branes form a circle. This means that $N_c - \tilde{N}_c$ magnetic monopoles (or dyons), which are between

adjacent color branes, become massless and can acquire vevs. The baryonic branch has one more massless monopole compared to the nonbaryonic branch, which is enough to completely Higgs the $U(1)^{N_c - \tilde{N}_c}$, and therefore there are enough vevs to control the center of mass motion of this set of color branes. The center of mass mode of the entire system is infinitely massive and so a shift in the center of mass of the $N_c - \tilde{N}_c$ branes leads to an opposite center of mass shift of the Seiberg dual $SU(\tilde{N}_c)$ gauge theory along the w plane, the same shift parametrized by all quark vevs. One result is that the magnetic monopoles are charged under the $U(1)_B$ baryon number. The crucial implication is that a slide along the w plane in the $U(1)^{N_c - \tilde{N}_c}$ system can undo the w -shift in the $SU(\tilde{N}_c)$ configuration which led to the flavor symmetry breaking quark vev. Thus flavor symmetry is unbroken on the baryonic branch.

These geometrical quantities can be related to the corresponding field theory calculation [42, 43, 54] by considering the following superpotential terms [24] in the magnetic description of the baryonic root

$$W \supset \frac{1}{\tilde{N}_c} \text{Tr}(q\tilde{q}) \left(\sum_{k=1}^{N_c - \tilde{N}_c} \psi_k \right) - \sum_{k=1}^{N_c - \tilde{N}_c} \psi_k e_k \tilde{e}_k + \mu \Lambda \sum_{k=1}^{N_c - \tilde{N}_c} x_k \psi_k. \quad (2.25)$$

Here q and \tilde{q} are magnetic quarks, e_k and \tilde{e}_k are flavorless magnetic monopoles, x_k are constants and ψ_k are the dual photons of the abelian part of the $SU(\tilde{N}_c) \times U(1)^{N_c - \tilde{N}_c}$ dual gauge group. From the superpotential we see that each magnetic monopole is charged under a $U(1)$ while the dual quarks are charged under all of the $U(1)$'s.⁵ As a result the D term equation for each $U(1)$ is

$$\frac{1}{\tilde{N}_c} \text{Tr}(q\tilde{q}) - e_k \tilde{e}_k + \mu \Lambda x_k = 0. \quad (2.26)$$

This means that, because there are as many e_k 's as x_k 's, $\text{Tr}(q\tilde{q})$ can vanish if each $e_k \tilde{e}_k$ is chosen correctly and, in fact, this solution is consistent with the rest of the D and F term constraints. Thus we see that the D term equation for the difference of two vevs is

⁵This is in contrast to the nonbaryonic root, where there is one less massless magnetic monopole and so after a basis change the dual quarks and magnetic monopoles are charged under disjoint $U(1)$'s.

interpreted in M-theory as the following statement. If the connected components of the M5-brane slide apart in the w plane along the $D6$ branes bisecting the $\tilde{N}_c - r$ color branes then, because the components are rigid, they also separate along the w plane at the semi-infinite flavor branes. In other words by trading a monopole vev for a magnetic quark vev, we find vacua which preserve flavor symmetry. These two distances, whose differences are preserved, are marked with double-headed arrows in Fig. 2.15.

To count vacua on this branch, consider the dual $SU(\tilde{N}_c)$ theory whose gauge symmetry is broken by the adjoint scalar vevs of the r color branes attached to flavor branes. As in the semiclassical case, the remaining $\tilde{N}_c - r$ color branes form a line which can have $\tilde{N}_c - r$ orientations preserving the M theory coordinate asymptotically far away, yielding a multiplicity of $\tilde{N}_c - r$ times the combinatoric factor of $\binom{N_f}{r}$ from the choice of which flavor branes to connect. In all, this provides

$$\mathcal{N}_2 = \sum_{r=0}^{\tilde{N}_c} (\tilde{N}_c - r) \binom{N_f}{r} = \mathcal{N}_{sc} - \mathcal{N}_1 \quad (2.27)$$

states, and thus completes the classification of vacua.

2.4.6 An example, $SU(3)$ with $N_f = 4$

We will consider in some detail the M5-branes associated with the r -vacua which survive breaking to $\mathcal{N} = 1$ for the case of $SU(3)$ with $N_f = 4$ and equal quark masses, $m_i = m$. This case is rather special in that the coordinates of these vacua in the moduli space can be analytically determined as a function of mass. Thus, we can explicitly draw out the corresponding M5-brane configurations and test our claims for the brane interpretations of the r vacua in various regimes.

In the case of equal and large bare quark masses, we can clearly see that the M5-brane configurations which survive breaking to $\mathcal{N} = 1$ in Fig. 2.16 do indeed correspond to r flavor branes and color branes connected to each other. In the massless case, the relevant curves are more difficult to interpret, though we include them for completeness in Fig. 2.17. Note that in the limit that $m \rightarrow 0$, the $r = 1$ non-baryonic root and the baryonic root converge

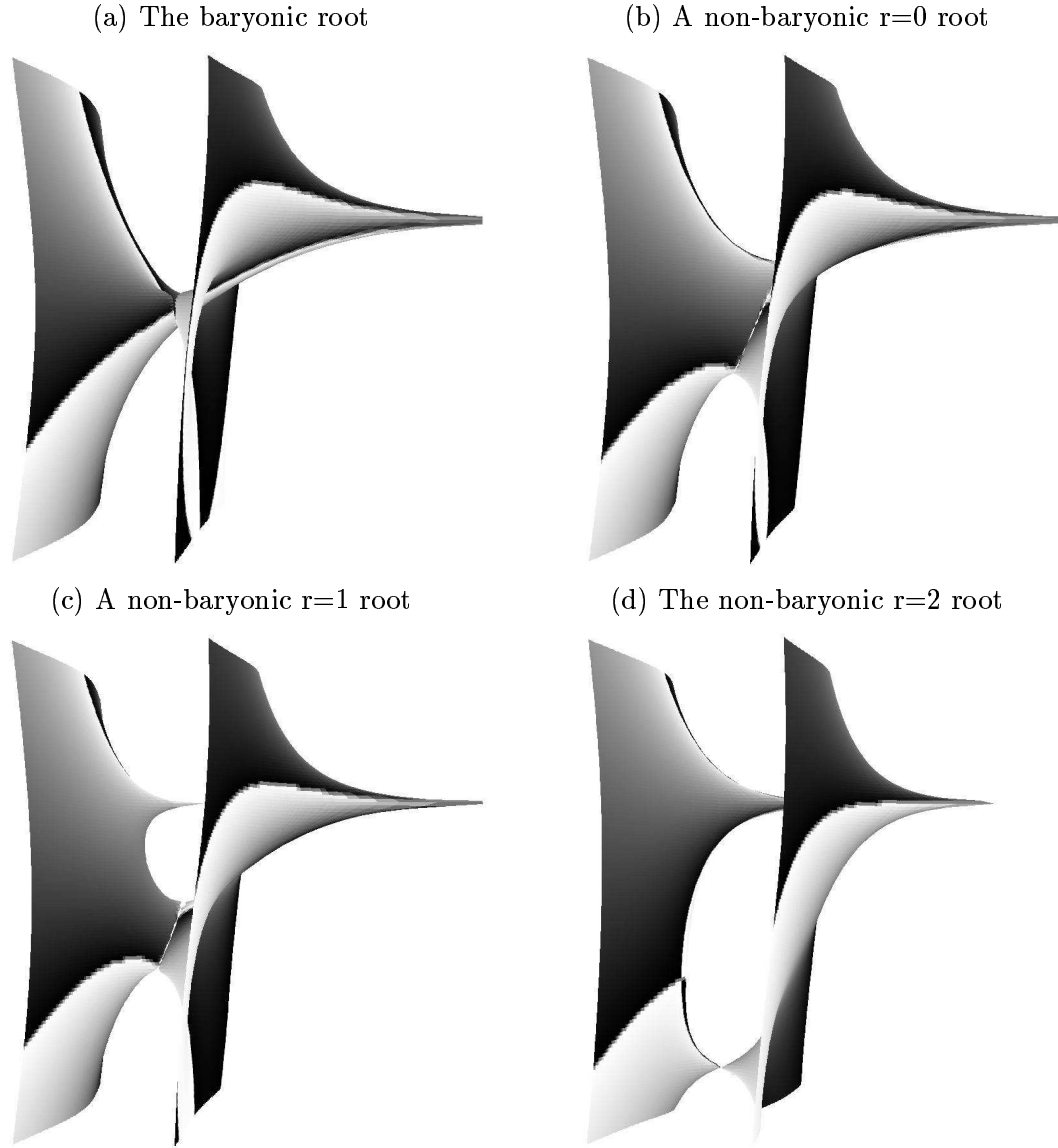


Figure 2.16: The M5-brane configurations corresponding to r -vacua in $SU(3)$ with 4 flavors in the semiclassical limit, i.e. with $m_i = m \gg \Lambda$. The flavor branes and color branes which have connected are represented by semi-infinite tubes extending from the left branch of the M5-brane. The connections between the two NS5-branes are the remaining color branes, pairs of which have condensed massless monopoles between them.

to the same point in moduli space, the $m = 0$ baryonic root. Just as in the superconformal case, it is easier to interpret these brane configuration if we consider their cross-sections in Figs. 2.18 and 2.19, though we will leave the interpretation of these cross-sections for future work.

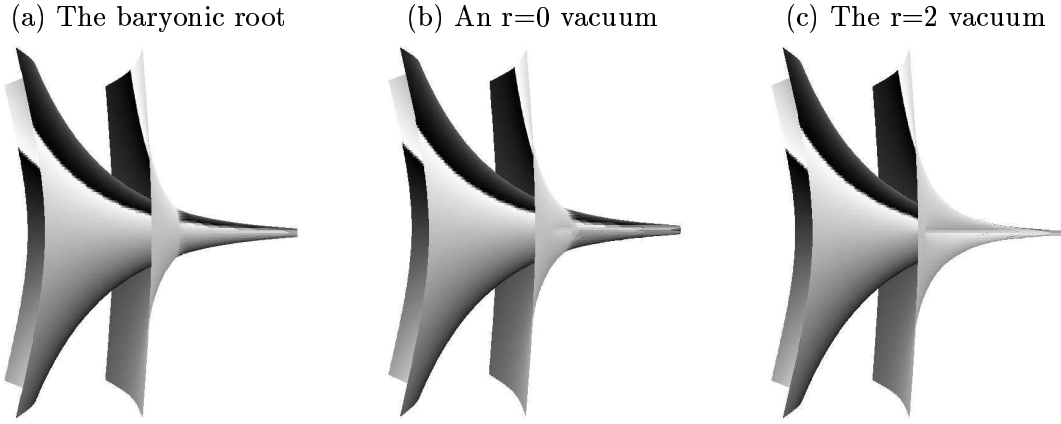


Figure 2.17: M5-brane configurations at the r vacua in $SU(3)$ with 4 flavors and $\forall m_i = 0$.

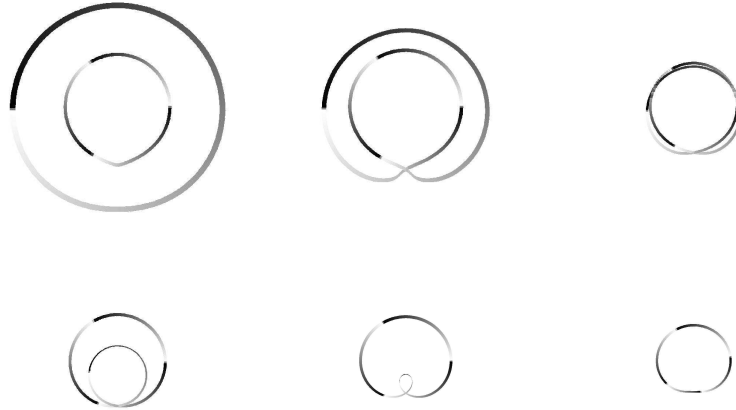


Figure 2.18: Cross sections at constant x^6 of the M5-brane of Fig. 2.17b (the $r=0$ root).



Figure 2.19: Cross sections at constant x^6 of the M5-brane of Fig. 2.17a (the baryonic root).

2.5 Discussion

Using the duality of type IIA string theory and M-theory, we have gained a geometric window on non-perturbative physics in supersymmetric gauge theories. We have constructed a new realization of Seiberg duality that relies on an energy scale dependent reduction of M-theory to IIA. We have found the M2-branes that correspond to flavored magnetic monopoles and argued that they correspond to magnetic baryons in the dual magnetic theory, which in turn decay to magnetic quarks. And finally we have interpreted baryon vevs as the relative sliding of two halves of an M theory configuration along a Taub-NUT singularity.

As an application of the above constructions, we have reproduced the field theory results of [43]. In particular we have correctly reproduced the flavor symmetry breaking patterns, the order parameters of the symmetry breaking and the counting of states in various regimes. We can interpret these countings in terms of discrete rotations of a line of M5-brane in the v plane.

For future work, we note that the case $r = N_f/2$ is difficult to analyze using traditional field theory techniques, as it is superconformal and strongly coupled in the IR. However it is possible that by deforming the corresponding curve, an analysis similar to the Seiberg duality of monopoles above may be possible in this M-theory setting.

Thus, we see that a modified classical geometry which incorporates the non-perturbative

behaviour of D0-brane probes through an extra dimension provides a powerful tool for analyzing gauge theories. One might wonder if the behaviour of D-branes in string theory can result in descriptions of physics which depart from the naive classical geometry in more substantial ways. The next section explores a circumstance in which non-commutative geometry of space-time arises naturally as the effective description of certain brane world-volume theories in string theory.

Chapter 3

Noncommutative Gauge Theories and the Seiberg-Witten Map

As we have noted earlier, the fact that the fluctuations of N coincident D-branes have a description in terms of a theory of $N \times N$ matrix valued fields is suggestive of a role for a noncommutative generalization of geometry [45] in string theory. By noncommutative geometry, we simply mean the possibility that the coordinates of space-time are no longer commuting observables. The simplest such example would be an imitation of the noncommutativity of position and momenta in quantum mechanics, but for space-time coordinates,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (3.1)$$

where $\theta^{\mu\nu}$ is very small. Note that the non-trivial commutation relation between the space-time coordinates imply that $\theta^{\mu\nu}$ defines a minimal quantum of area.¹ Therefore, one would expect that quantum field theories defined on such a space-time would be intrinsically non-local. As to the origin of the noncommutativity, one might imagine that $\theta^{\mu\nu}$ arose, in analogy with the Higgs effect, as the vacuum expectation value of an antisymmetric tensor field. However, as $\theta^{\mu\nu}$ is a tensor rather than a scalar, its components give rise to

¹Note that such quantization of area, though of a different character, can also be motivated by studies of quantum gravity as seen explicitly in the work of Ashtekar and collaborators [8].

a preferred direction in space-time. This preferred direction in any given frame violates “particle Lorentz symmetry” (see [56, 57] for a review), and suggests that limits on $\theta^{\mu\nu}$ may be obtained by comparing with a variety of Lorentz violation experiments.

However, to address these issues and make concrete predictions certainly requires that one can define quantum field theories on a noncommutative space-times at all! Due to the presence of non-locality, the possibility of making sense of such theories was regarded with skepticism by much of the physics community. The situation changed dramatically when it was realized that a generalization of $U(N)$ Yang-Mills gauge theories defined on noncommutative spaces appears (Noncommutative Super Yang-Mills, NCSYM) in a certain limit of string theory [58, 59] on the worldvolume of D-branes. Further, as we will review shortly, it was noted [59] that in that limit, a conventional description in terms a gauge theory on a commutative space was still possible, though with infinitely many higher derivative couplings proportional to powers of $\theta^{\mu\nu}$ added to the action. Therefore, they argued that there must exist a mapping between commutative gauge field configurations to noncommutative ones which is compatible with the gauge structure of each. The mapping between these descriptions has come to be known as the Seiberg-Witten (SW) map. Describing a method for explicitly computing this map order by order in θ for any gauge group G is the primary focus of this chapter. One of the primary uses of the Seiberg-Witten map is that it allows one to concretely analyze theories on noncommutative spaces as theories defined on commutative spaces with additional higher dimensional couplings. Further [63], the map also provides a method of constructing noncommutative gauge theories with gauge groups other than $U(n)$.

We begin with a review of the relevant facts regarding gauge theories on noncommutative spaces and the physics underlying the Seiberg-Witten map, as well as some basic facts regarding expansions of the star product that will be useful in the rest of the paper. In section 3.2, we review the methods developed in [63], which provide an essential starting point for our work. In Section 3.3 we replace the gauge parameters appearing in the SW map with a ghost field, which makes explicit a cohomological structure underlying the SW map. In Section 3.4 we define a homotopy operator, which can be used to explicitly write

down the SW map order by order in θ . Finally, in Section 3.5 we apply our methods to calculate some low order terms of the SW map. Note that the results that follow were obtained in collaboration with Dan Brace, Bianca Cherchia, Andrea Pasqua, and Bruno Zumino [74].

3.1 General Review of NCSYM in String Theory

3.1.1 Definition of NCYM

We begin with a quick overview of the standard definition of noncommutative Yang-Mills (NCYM) theories in \mathbb{R}^{2r} . The crucial construction that we will use to define a field theory on noncommutative \mathbb{R}^{2r} is the Moyal-Weyl product. This product, the “ \star ” product, is a noncommutative and associative deformation of the usual, commutative product on the space of functions on \mathbb{R}^{2r} associated with a *constant* Poisson tensor θ^{ij} , and is defined by,

$$f \star g = f \frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j g . \quad (3.2)$$

In particular, this product has the property that it implements the noncommutativity of the coordinate functions we alluded to above,

$$[x^i \star x^j] = i\theta^{ij}. \quad (3.3)$$

In fact, we take this deformed algebra of functions on ordinary \mathbb{R}^{2r} as defining what we mean by noncommutative \mathbb{R}_θ^{2r} . The primary utility of the Moyal-Weyl product is that it allows us to define field theories on noncommutative spaces by using the actions, gauge symmetries, etc. of the corresponding theory on the commutative space, and just replacing ordinary products with star products. So, we can define noncommutative gauge fields A_i which tranform under infinitesimal noncommutative gauge transformations generated by Λ as,

$$\hat{\delta} A_i = \partial_i \Lambda + i\Lambda \star A_i - iA_i \star \Lambda. \quad (3.4)$$

We define the noncommutative field strength associated with this gauge field to be,

$$F_{ij} = \partial_i A_j - \partial_j A_i - iA_i \star A_j + iA_j \star A_i. \quad (3.5)$$

Now, as we will explain in the next section, one finds that string theory defines an effective notion of a metric G_{ij} on the noncommutative \mathbb{R}^{2r} that can be used to construct an action for this noncommutative gauge theory,

$$S = \int \sqrt{\det G} G^{ii'} G^{jj'} \text{Tr } F_{ij} \star F_{i'j'}. \quad (3.6)$$

One can, in principle, use the above action to quantize the theory perturbatively, with the only modifications being non-local phases associated with the substitution of Moyal-Weyl products rather than ordinary products at vertices. We will not pursue this line of reasoning further here but note that the resulting perturbative expansion has subtleties [60] associated with infrared divergences. For example, when θ^{ij} is large, the Moyal phases make the noncommutative perturbative expansion more tractable, but one finds subtleties due to light $M^2 \sim \theta^{-1}$, large $l \sim \sqrt{\theta}$ solitons and instantons. In particular, even though though the theory is formally very similar to ordinary Yang-Mills, it is not clear that one can use the usual arguments of perturbative QFT to show that this theory is well defined. However, arguments from string theory suggest that at least for supersymmetric cousins of the above theory, this is in fact the case.

3.1.2 NCSYM from String Theory

To gain intuition for how noncommutativity might arise in string theory, it is useful to review an very familiar quantum mechanical system where noncommutativity of space-time coordinates appears - the motion of an electron in a plane moving in a large transverse magnetic field,

$$L = \frac{\dot{x}^2}{2m} + \frac{\dot{y}^2}{2m} - \frac{eB}{c} xy. \quad (3.7)$$

Since $p_x = m\dot{x}$ and $p_y = m\dot{y} - \frac{eB}{c}x$, we see that as $B \rightarrow \infty$, the kinetic energy can be neglected and x and y become canonically conjugate. In the quantum theory, this limit is equivalent to making a projection P onto to the lowest Landau level, after which the operators $Px^iP \approx x^i$ no longer commute

$$[x, y] = i \frac{c}{eB}. \quad (3.8)$$

This suggests that considering the second quantization of such a system might be interpretable as a field theory defined on a *noncommutative* space. In string theory, the end-points of open strings on a D-brane act like charges (i.e. the W -bosons of the gauge theory). Thus, perhaps noncommutativity would appear on a D-brane if one adds a large magnetic field. It turns out that a non-zero NS B -field acts precisely like a magnetic field on the world-volume of the brane. This can be seen by noting that the gauge symmetry of the NS B -field $B \rightarrow B + d\Lambda_{NS}$ also acts on the center of mass $U(1)$ part of the gauge field of the brane $a \rightarrow a - \Lambda_{NS}$, so that gauge invariant quantities must always be constructed out of the quantity $f + B$. In particular, one can trade a non-zero B -field on a brane for a magnetic field strength of the center of mass $U(1)$ via a gauge transformation.

Now, we review the work of Seiberg and Witten [59]. Consider open string theory with worldsheet Σ and boundary $\partial\Sigma$ mapping to the worldvolume of the brane. While the arguments we present involve only the bosonic fields on the worldsheet and therefore could be done for the bosonic string, we restrict ourselves to the non-tachyonic superstring, simply omitting the fermions for brevity. Assume the space-time is flat, with the sigma model, classical closed string metrics g_{ij} and B_{ij} constant, where i, j are taken to run over just the *spatial* directions on the brane. Note that in what follows, we will only be considering the case of noncommutativity in a time-like direction. The action for the open string is,

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int_{\Sigma} \left(g_{ij} \partial_a x^i \partial^a x^j - 2\pi i \alpha' B_{ij} \epsilon^{ab} \partial_a x^i \partial_b x^j \right) \\ &= \frac{1}{4\pi\alpha'} \int_{\Sigma} g_{ij} \partial_a x^i \partial^a x^j - \frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j. \end{aligned} \quad (3.9)$$

A background gauge field (in the $U(1)$ case) $a_i(x)$ on the brane couples to the string worldsheet by

$$-i \int d\tau a_i(x) \partial_\tau x^i. \quad (3.10)$$

Now it is easy to see that a constant B -field on the brane has the same physical effect as $a_i = -\frac{1}{2} B_{ij} x^j$, i.e. magnetic field $f = B$. Further, note that the effect of B_{ij} , note that

consider the is to consider the boundary conditions of the open strings,

$$g_{ij}\partial_n x^j + 2\pi i\alpha' B_{ij}\partial_t x^j|_{\partial\Sigma} = 0, \quad (3.11)$$

Thus, we see that increading B_{ij} interpolates between Neumann and Dirichlet boundary conditions on pairs of directions along the brane. Thus, the Dp-brane appears to break up into an infinite collection of D(p-2r) branes, where 2r are the directions along which B_{ij} is taken large. If B is large in all spatial directions, the Dp-brane becomes an infinite collection of D0-branes. The low energy effective theory that describes coincident D0-branes is the dimensional reduction of SYM in $9 + 1D$ down to $0 + 1D$ matrix quantum mechanics with gauge group $U(\infty)$. Now, in the last section, we briefly reviewed how one can construct Yang-Mills theory on a noncommutative space using functions on commutative \mathbb{R}^n with a deformed Moyal-Weyl product. However, one can instead use the matrix representation of the commutation relations 3.1 on an auxiliary Hilbert space. The Yang-Mills fields are then just operators acting on the Hilbert space, and the action is constructed using an appropriately defined trace over the Hilbert space. One can in fact show that the resulting formulation of the supersymmetric, noncommutative Yang-Mills theory [61] is precisely the $U(\infty)$ gauged matrix theory associated with the constituent D0-branes described above.

While this approach provides a beautiful interpretation of the origin of noncommutative geometry in terms of the non-abelian nature of the gauge theory on D0-branes, a more direct construction of NCSYM in terms of a Moyal-Weyl formulation is desirable. Towards that end, let us consider very explicitly the construction of a low energy effective theory for the open string theory at tree-level. Consider open string worldsheets which look like several strips extending out to infinity (associated asymptotic open string states) attached at a disk (representing the smeared out interaction vertex). These worldsheets represent the lowest order contribution to the scattering of the associated asymptotic string states. As the string worldsheet theory is conformally invariant, one can use that fact that there exists a conformal mapping of this worldsheet to a compact disk to simplify matters. In particular, this transformation maps the ends of each of the infinite strips (which encode the asymptotic open string state) to points on the boundary of the disk. To each of these

points is associated an operator which reflects the asymptotic string state, known as a vertex operator. Therefore, the correlation functions of operators inserted on the boundaries of the disk can be used to compute the interactions of asymptotic open string states. The vertex operators for strings are constructed out of the fields x^i which describe the embedding of the string into the space-time. States of momentum p in space-time correspond to vertex operators which have factors of the form $e^{ip \cdot x}$. Thus, in order to compute interactions between such states, we need to first compute the correlation functions of the x^i fields on the disk with the boundary conditions associated with the presence of the non-zero NS B -field. This is most easily done by making an additional conformal mapping to the upper half plane, $\text{Im } z = 0$, in which the boundary conditions become,

$$g_{ij}(\partial - \bar{\partial})x^j + 2\pi i \alpha' B_{ij}(\partial - \bar{\partial})x^j \Big|_{z=\bar{z}} = 0. \quad (3.12)$$

The correlation functions with these boundary conditions have been computed in [62], yielding,

$$\langle x^i(z) x^j(z') \rangle = -\alpha' [g^{ij} \log |z - z'| - g^{ij} \log |z - \bar{z}'|] \quad (3.13)$$

$$+ G^{ij} \log |z - \bar{z}'|^2 + \frac{1}{2\pi\alpha'} \theta^{ij} \log \frac{z - \bar{z}'}{\bar{z} - z'} + D^{ij}, \quad (3.14)$$

where we have defined

$$G^{ij} = \left(\frac{1}{g + 2\pi\alpha' B} \right)_S^{ij} = \left(\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B} \right)^{ij}, \quad (3.15)$$

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B} \right)_A^{ij} = -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B g - 2\pi\alpha' B \right)^{ij}, \quad (3.16)$$

and the D^{ij} are unimportant for what follows. Closed strings correspond to operator insertions in the bulk of the disk, and one can easily see that singularities associated with closed string operators arise only from the first two terms. These singularities are responsible for anomalous dimensions of the vertex operators $e^{ip \cdot x}$ which give rise to the mass shell condition of the corresponding states. Thus, it is natural to associate the metric g^{ij} with closed strings. As asymptotic open strings correspond to operator insertions on the

boundary, to perform the same analysis for the open strings, we need the restriction of the above expression to the fields on the boundary (here $\epsilon(\pm) = \pm 1$),

$$\langle x^i(\tau) x^j(\tau') \rangle = -\alpha' G^{ij} \log(\tau - \tau')^2 + \frac{i}{2} \theta^{ij} \epsilon(\tau - \tau'), \quad (3.17)$$

We see that open string mass shell conditions are likewise associated with the metric G^{ij} . Thus, it is natural to consider G^{ij} as the effective metric for open strings. Now, what is the effect of θ^{ij} ? It just makes the end-points of open strings noncommutative,

$$[x^i(\tau), x^j(\tau)] = T (x^i(\tau) x^j(\tau^-) - x^i(\tau) x^j(\tau^+)) = i \theta^{ij}. \quad (3.18)$$

To see its effect on the scattering amplitudes of asymptotic open string states, we use the above correlation functions to construct the tree level scattering of k such open strings,

$$\left\langle \prod_{n=1}^k e^{ip^n \cdot x(\tau_n)} \right\rangle_{G, \theta} = e^{-\frac{i}{2} \sum_{n>m} p_i^n \theta^{ij} p_j^m \epsilon(\tau_n - \tau_m)} \times \left\langle \prod_{n=1}^k e^{ip^n \cdot x(\tau_n)} \right\rangle_{G, \theta=0}. \quad (3.19)$$

Thus, we see that the only effect of θ^{ij} is to introduce certain additional phases into the open string scattering amplitudes - in fact, it is easy to see that these phases are just associated with the Fourier transform of the Moyal-Weyl deformation of the ordinary product! Now, in string theory, each open string state corresponds with a field on the brane and k -point tree level amplitudes are associated with the couplings of the product of k such fields in the action. Thus, we can model the effect of θ as (at tree level in string coupling) by deforming the associated coupling in the action by

$$\int Tr \Phi_1 \Phi_2 \cdots \Phi_n \rightarrow \int Tr \Phi_1 \star \Phi_2 \star \cdots \star \Phi_n. \quad (3.20)$$

Therefore, we find that the effective action of this theory is naturally expressed in terms of \star products, and can therefore be interpreted as a theory defined on a noncommutative space. However, we would like to exhibit a limit in which the physics is completely captured by pure NCSYM. To do this, we need to take the low energy limit and decouple massive open and closed string modes by taking $\alpha' \rightarrow 0$ keeping G^{ij} and θ^{ij} fixed,

$$\alpha' \sim \epsilon^{\frac{1}{2}} \rightarrow 0 \quad g_{ij} \sim \epsilon \rightarrow 0 \quad \text{for } i, j = 1, \dots, 2r, \quad (3.21)$$

where $i = 1, \dots, 2r$ are the spatial directions along the brane. In this limit (the SW limit), $\langle x^i(\tau)x^j(0) \rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau)$, and for any pair of vertex operators representing fields on the brane, $f(x(\tau))$ and $g(x(\tau))$, we have the operator product

$$\lim_{\tau \rightarrow 0^+} : f(x(\tau)) : : g(x(0)) := f(x(0)) \star g(x(0)) : . \quad (3.22)$$

Thus, in this limit, the algebra of vertex operators and therefore the space of fields in the effective action on the brane is explicitly deformed into a noncommutative algebra.

However, while we have seen that the effective action can be naturally expressed in terms of deformed products, we have no reason yet to expect that the gauge symmetries of the theory are also deformed. Thus, we need to analyze the gauge symmetries of these fields in the SW limit. Gauge symmetries in space-time are associated with symmetries of the world sheet path integral which are implemented by a BRST physical state condition. Clearly, $\delta_\lambda a_i = \partial_i \lambda$ is manifestly a symmetry of the classical worldsheet action, even with modified boundary conditions. However, we must check that the full quantum theory does not violate this symmetry. In particular, we need to consider the variation $\delta_\lambda a_i = \partial_i \lambda$ of the full string path integral by expanding $e^{-i \int d\tau a_i(x) \partial_\tau x^i}$ in powers of a_i and varying,

$$\delta_\lambda Z = -i \int d\tau \partial_\tau \lambda - \int d\tau a_i(x) \partial_\tau x^i \cdot \int d\tau' \partial_{\tau'} \lambda \quad (3.23)$$

The subtlety here is that the second term contains operators evaluated at the same worldsheet points, and therefore have infinite ambiguities associated with them. Their value depends on a choice of regularization of the worldsheet path integral. Pauli-Villars regularization can be used to show that ordinary gauge symmetry is a possibility. However, if we consider point-splitting regularization in which we never allow operators to come together on the boundary by cutting out the regions $|\tau - \tau'| < \delta$ and taking $\delta \rightarrow 0$, we get

$$\begin{aligned} \delta_\lambda Z &= -i \int d\tau \partial_\tau \lambda - \int d\tau : a_i(x(\tau)) \partial_\tau x^i(\tau) :: (\lambda(x(\tau^-)) - \lambda(x(\tau^+))) : \\ &= -i \int d\tau \partial_\tau \lambda - \int d\tau : (a_i(x) \star \lambda - \lambda \star a_i(x)) \partial_\tau x^i : . \end{aligned} \quad (3.24)$$

Thus, in this regularization scheme, the symmetry transformation must be deformed to

(taking $\lambda \rightarrow \Lambda$),

$$\hat{\delta} A_i = \partial_i \Lambda + i\Lambda \star A_i - iA_i \star \Lambda. \quad (3.25)$$

This is the gauge invariance of (a $U(1)$) noncommutative Yang-Mills theory. Note that the group of noncommutative gauge transformations in the $U(1)$ case is non-abelian. Further, in the SW limit, one can explicitly compute the three point functions for gauge boson scattering and find that it is generated by a term in the effective action proportional to,

$$\int \sqrt{\det G} G^{ii'} G^{jj'} \text{Tr } F_{ij} \star F_{i'j'}, \quad (3.26)$$

This is the action of NCYM. One can check that this action exhibits the noncommutative gauge invariance. Therefore, in the SW limit, we see that if one uses point splitting regularization of the worldsheet theory, the effective action can naturally expressed in terms of fields which have a noncommutative gauge invariance. However, if we had instead chosen Pauli-Villars regularization, the gauge symmetry of the effective theory on the brane would be undeformed. Now, different choices of regulators on the worldsheet differ by coupling constant redefinitions on the worldsheet. Since coupling constants on the worldsheet correspond to spacetime fields, this means that the commutative and noncommutative gauge theory descriptions must be related by a field redefinition. This field redefinition is the Seiberg-Witten Map.

3.1.3 Formal Properties of the SW Map

Let us analyze more carefully the properties such a field redefinition must possess. First note that the physical equivalence of the two descriptions only requires that the *physical* configuration spaces of the commutative and noncommutative theory coincide. In particular, this means that one only needs to have an equivalence (at least locally) between the spaces of *gauge orbits* of the two theories. Thus, the Seiberg-Witten map must be a field redefinition that transforms commutative gauge fields a_i to commutative ones A_i and maintains gauge equivalence. So, if two field configurations were gauge equivalent in one picture, they must map to gauge equivalent configurations in the other. But the gauge

groups are not isomorphic. Thus, the map cannot be of the form $A_i(a_i)$, with infinitesimal commutative gauge transformations mapped to noncommutative ones $\Lambda(\lambda)$. So, we allow $\Lambda(\lambda, a_i)$, and demand the following gauge equivalence condition holds,

$$A(a) + \delta_{\Lambda(a, \lambda)} A = A(a + \delta_\lambda a), \quad (3.27)$$

where we assume that λ is infinitesimal and expand $A(a) = a + A'(a)$, $\Lambda(a, \lambda) = \lambda + \Lambda'(a, \lambda)$ where A' and Λ' are taken to be power series in θ . One can expand the above condition order by order in θ^{ij} and attempt to look for solutions perturbatively in θ . Seiberg and Witten found that the following expressions,

$$A_i(a) = a_i - \frac{1}{4} \theta^{kl} \{a_k, \partial_l a_i + f_{li}\} + \mathcal{O}(\theta^2), \quad (3.28)$$

$$\Lambda(\lambda, a) = \lambda + \frac{1}{4} \theta^{ij} \{\partial_i \lambda, a_j\} + \mathcal{O}(\theta^2), \quad (3.29)$$

satisfy the above condition to first order. However, as was first noted by Asakawa et al. in [71], this solution is not unique. The Seiberg-Witten map contains ambiguities, even to first order. In particular, one can add to Λ

$$\tilde{\Lambda}^{(1)} = -2i\theta^{ij} [\partial_i \lambda, a_j], \quad (3.30)$$

to find another solution to first order. Further, Seiberg and Witten solved the map explicitly for the case of constant field strength, and found

$$F = \frac{1}{1 + f\theta} f, \quad (3.31)$$

which is singular when $f = -\theta^{-1}$. That is, when the magnetic field due to the gauge field and the B field cancel, the noncommutative description fails. This is not surprising as in this case, both can be gauged away to zero. However, it does suggest that the map should not be expected to exist globally, and should only be sensible in a systematic long wavelength expansion.

Nevertheless, it is worthwhile attempting to understand the global mathematical structure of this map more formally.² Let \mathcal{G} and \mathcal{G}_θ be the ordinary and deformed infinite

²The following considerations are very similar to arguments in [65].

dimensional groups of gauge transformations, and \mathcal{A} the affine space of connection one forms. Globally, the SW map should be a pair of (possibly singular) maps

$$A_\theta : \mathcal{A} \rightarrow \mathcal{A}, \quad U_\theta : \mathcal{A} \times \mathcal{G} \rightarrow \mathcal{G}_\theta \quad (3.32)$$

which descends to an “isomorphism”, $\mathcal{A}/\mathcal{G} \approx \mathcal{A}/\mathcal{G}_\theta$. This suggests a global definition of the SW map. For $g, g' \in \mathcal{G}$, define

$$g(a) = g^{-1}ag + g^{-1}dg. \quad (3.33)$$

We propose that a SW map obeys,

1. $A_\theta(g(a)) = U_\theta(a, g)^{-1} \star A_\theta(a) \star U_\theta(a, g) + U_\theta(a, g)^{-1} \star dU_\theta(a, g)$
2. $U_\theta((gg')(a), gg') = U_\theta(g'(a), g) \star U_\theta(a, g')$
3. $U_\theta(g^{-1}(a), g) = U_\theta(a, g^{-1})^{-1}$.
4. $U_\theta((g'g'')(a), g) \star U_\theta(a, g'g'') = U_\theta(g''(a), gg') \star U_\theta(a, h)$

The star products above are meant to denote the deformed group multiplication law (both a \star and group multiplication). As we noted above, the map is far from unique. One should get an equivalent map by first applying a fixed gauge transformation to a and then applying the SW map. More precisely, for some $h \in \mathcal{G}$, consider $U'(a, g) \equiv U_\theta(h^{-1}ah + h^{-1}dh, g)$ and $A'(a) \equiv A_\theta(h^{-1}ah + h^{-1}dh)$. One can show that if A and U satisfy the above conditions then so do A' and U' . Along the same lines, one is also free to add to a any global adjoint valued two form before applying the SW map. These can be constructed using covariant derivatives, θ 's and F 's, thereby making use of the affine structure of \mathcal{A} . These translate to the aforementioned ambiguities in the infinitesimal version of the map which were the original motivation the cohomological approach which follows.³ As was pointed out by A. Weinstein, the conditions outlined above are precisely those of a groupoid morphism between action groupoids associated with action of the commutative and noncommutative

³The infinitesimal version of these ambiguities were first noted by R. Stora and examined in detail in [75].

gauge transformations on the space of connections. Further, Weinstein was able to show that the infinitesimal version of this transformation could be understood as a Lie algebroid morphism. Mathematically, the aim of this work is to find explicit expressions for this Lie algebroid morphism.

3.2 The Wess Formalism for the SW Map

In this section, we review the formalism developed in [63], which provides an alternative method for obtaining an expression for the SW map. To recap, recall that the condition which defines the SW map [59] was obtained by the requirement that the infinitesimal gauge transformations of the commutative gauge field a_i under λ and the noncommutative gauge field A_i under Λ ,

$$\delta_\lambda a_i = \partial_i \lambda - i[a_i, \lambda], \quad (3.34)$$

$$\delta_\Lambda A_i = \partial_i \Lambda - i[A_i \star \Lambda] \equiv \partial_i \Lambda - i(A_i \star \Lambda - \Lambda \star A_i). \quad (3.35)$$

be related by,

$$A_i + \delta_\Lambda A_i = A_i(a_j + \delta_\lambda a_j, \dots). \quad (3.36)$$

Now, in order to satisfy (3.36) the noncommutative gauge field and gauge parameter must have the following functional dependence.

$$\begin{aligned} A_i &= A_i(a, \partial a, \partial^2 a, \dots) \\ \Lambda &= \Lambda(\lambda, \partial \lambda, \dots, a, \partial a, \dots), \end{aligned} \quad (3.37)$$

where the dots indicate higher derivatives. As we indicated in the last section, a SW map is not uniquely defined by condition (3.36). The ambiguities that arise [71] will be discussed shortly.

The condition (3.36) yields a simultaneous equation for A_i and Λ . For the constant θ case, explicit solutions of the Seiberg-Witten map have been found by various authors up to second order in θ [72, 63]. The solutions were found by writing the map as a linear combination of all possible terms allowed by index structure and dimensional constraints

and then determining the coefficients by plugging this expression into the SW equation. The method we will describe in the rest of the paper provides a more systematic procedure for solving the SW map. For the special case of a $U(1)$ gauge group, an exact solution in terms of the Kontsevich formality map is given in [64], while [66, 69, 68, 69] present an inverse of the SW map to all orders in θ .

An alternative characterization of the Seiberg-Witten map can be obtained following [63]. In the commutative gauge theory, one may consider a field ψ in the fundamental representation of the gauge group. If we assume that the SW map can be extended to include such fields, then there will be a field Ψ in the noncommutative theory with the following functional dependence

$$\Psi = \Psi(\psi, \partial\psi, \dots, a, \partial a, \dots), \quad (3.38)$$

and with the corresponding infinitesimal gauge transformation

$$\delta_\lambda \psi = i\lambda \psi \quad (3.39)$$

$$\delta_\Lambda \Psi = i\Lambda \star \Psi. \quad (3.40)$$

An alternative to the SW condition (3.36) can now be given by

$$\Psi + \delta_\Lambda \Psi = \Psi(\psi + \delta_\lambda \psi, \dots, a_j + \delta_\lambda a_j, \dots). \quad (3.41)$$

More compactly, one writes

$$\delta_{\Lambda_\lambda} \Psi(\psi, a_j, \dots) = \delta_\lambda \Psi(\psi, a_j, \dots). \quad (3.42)$$

The dependence of Λ on α is shown explicitly on the left hand side, and on the right hand side δ_λ acts as a derivation on the function Ψ , with an action on the variables ψ and a_i given by (3.39) and (3.34) respectively. Next, one considers the commutator of two infinitesimal gauge transformations

$$[\delta_{\Lambda_\lambda}, \delta_{\Lambda_\gamma}] \Psi = [\delta_\lambda, \delta_\gamma] \Psi. \quad (3.43)$$

Since $[\delta_\lambda, \delta_\gamma] = \delta_{-i[\lambda, \gamma]}$, the right hand side of (3.43) can be rewritten as

$$\delta_{-i[\lambda, \gamma]} \Psi = \delta_{\Lambda_{-i[\lambda, \gamma]}} \Psi = i\Lambda_{-i[\lambda, \gamma]} \star \Psi = \Lambda_{[\lambda, \gamma]} \star \Psi.$$

The last equality follows from the fact that Λ is linear in the ordinary gauge parameter, which is infinitesimal. As for the left hand side,

$$\begin{aligned} [\delta_{\Lambda_\lambda}, \delta_{\Lambda_\gamma}] \Psi &= \delta_{\Lambda_\lambda} (i\Lambda_\gamma \star \Psi) - \delta_{\Lambda_\gamma} (i\Lambda_\lambda \star \Psi) \\ &= i(\delta_\lambda \Lambda_\gamma - \delta_\gamma \Lambda_\lambda) \star \Psi + [\Lambda_\lambda \star \Lambda_\gamma] \star \Psi. \end{aligned}$$

Equating the two expressions and dropping Ψ yields

$$(\delta_\lambda \Lambda_\gamma - \delta_\gamma \Lambda_\lambda) - i[\Lambda_\lambda \star \Lambda_\gamma] + i\Lambda_{[\lambda, \gamma]} = 0. \quad (3.44)$$

An advantage of this formulation is that (3.44) is an equation in Λ only, whereas (3.36) must be solved simultaneously in Λ and A_i . If (3.44) is solved, (3.35) then yields an equation for A_i and (3.40) for Ψ .

3.3 The Ghost Field and the Coboundary Operator

Inspired by the simplicity of the BRST formulation of gauge theories, we rewrite equations (3.35), (3.40) and (3.44) in terms of a ghost field in order to make explicit an underlying cohomological structure. That is, we will redefine the gauge parameter λ as an anti-commuting, Grassmannian field which is enveloping algebra valued. Define a ghost number by assigning ghost number one to λ and zero to a_i and ψ . The ghost number introduces a Z_2 grading, with even quantities commuting and odd quantities anticommuting. In our formalism, the gauge transformations (3.34) and (3.39) are replaced by the following BRST transformations:

$$\begin{aligned} \delta_\lambda \lambda &= i\lambda^2 \\ \delta_\lambda a_i &= \partial_i \lambda - i[a_i, \lambda] \\ \delta_\lambda \psi &= i\lambda \psi. \end{aligned} \quad (3.45)$$

In the $U(1)$ case the introduction of a ghost was first considered in [73]. We also take δ_λ to commute with the partial derivatives,

$$[\delta_\lambda, \partial_i] = 0 . \quad (3.46)$$

The operator δ_λ has ghost number one and obeys a graded Leibniz rule

$$\delta_\lambda(f_1 f_2) = (\delta_\lambda f_1) f_2 + (-1)^{\deg(f_1)} f_1 (\delta_\lambda f_2) , \quad (3.47)$$

where $\deg(f)$ gives the ghost number of the expression f . One can readily check that δ_λ is nilpotent on the fields a_i , ψ and λ and therefore, as a consequence of (3.47), we have

$$\delta_\lambda^2 = 0 . \quad (3.48)$$

Following the procedure outlined in the previous section, we characterize the SW map as follows. We introduce a matter field $\Psi(\psi, \partial\psi, \dots, a, \partial a, \dots)$ and an odd gauge parameter $\Lambda(\lambda, \partial\lambda, \dots, a, \partial a, \dots)$ corresponding to ψ and λ in the commutative theory. Λ is linear in the infinitesimal parameter λ and hence has ghost number one. As before, we require that the SW map respect gauge invariance.

$$\delta_\Lambda \Psi \equiv i\Lambda \star \Psi = \delta_\lambda \Psi. \quad (3.49)$$

The consistency condition (3.43) now takes the form

$$\delta_\Lambda^2 \Psi = \delta_\lambda^2 \Psi = 0 , \quad (3.50)$$

and again it yields an equation in Λ only.

$$0 = \delta_\Lambda^2 \Psi = \delta_\Lambda (i\Lambda \star \Psi) = i\delta_\lambda \Lambda \star \Psi + \Lambda \star \Lambda \star \Psi$$

Since Ψ is arbitrary we obtain

$$\delta_\lambda \Lambda = i\Lambda \star \Lambda. \quad (3.51)$$

Once the solution of (3.51) is known, one can solve the following equations for Ψ and the gauge field.

$$\delta_\lambda \Psi = i\Lambda \star \Psi , \quad \delta_\lambda A_i = \partial_i \Lambda - i[A_i \star \Lambda] . \quad (3.52)$$

It is natural to expand Λ and A_i as power series in the deformation parameter θ . We indicate the order in θ by an upper index in parentheses

$$\begin{aligned}\Lambda &= \sum_{n=0}^{\infty} \Lambda^{(n)} = \lambda + \sum_{n=1}^{\infty} \Lambda^{(n)} \\ A_i &= \sum_{n=0}^{\infty} A_i^{(n)} = a_i + \sum_{n=1}^{\infty} A_i^{(n)} .\end{aligned}\tag{3.53}$$

Note that the zeroth order terms are determined by requiring that the SW map reduce to the identity as θ goes to zero. Using this expansion we can rewrite equations (3.51) and (3.52) as

$$\begin{aligned}\delta_\lambda \Lambda^{(n)} - i\{\lambda, \Lambda^{(n)}\} &= M^{(n)} \\ \delta_\lambda A_i^{(n)} - i[\lambda, A_i^{(n)}] &= U_i^{(n)} ,\end{aligned}\tag{3.54}$$

where, in the first equation, $M^{(n)}$ collects all terms of order n which do not contain $\Lambda^{(n)}$, and similarly $U_i^{(n)}$ collects terms not involving $A_i^{(n)}$. We refer to the left hand side of each equation as its homogeneous part, and to M and U_i as the inhomogeneous terms of (3.54). Note that $M^{(n)}$ contains explicit factors of θ , originating from the expansion of the Weyl-Moyal product (3.2). An expression for the generic $M^{(n)}$ is given in the Appendix. If the SW map for Λ is known up to order $(n-1)$, then $M^{(n)}$ can be calculated explicitly as a function of λ and a_i . On the other hand, $U^{(n)}$ depends on both Λ and A_i , the former up to order n and the latter up to order $(n-1)$. Still, one can calculate it iteratively as a function of λ and a_i .

The structure of the homogeneous parts of equation (3.54) suggests the introduction of a new operator Δ .

$$\Delta = \begin{cases} \delta_\lambda - i\{\lambda, \cdot\} & \text{on odd quantities} \\ \delta_\lambda - i[\lambda, \cdot] & \text{on even quantities} \end{cases}\tag{3.55}$$

In particular, Δ acts on λ and a_i as,

$$\Delta \lambda = -i\lambda^2 , \quad \Delta a_i = \partial_i \lambda .\tag{3.56}$$

As a consequence of its definition, Δ is an anti-derivation with ghost-number one. It follows a graded Leibniz rule identical to the one for δ_λ (3.47). Another consequence of the definition (3.55) is that Δ is nilpotent

$$\Delta^2 = 0 .\tag{3.57}$$

The action of Δ on expressions involving a_i and its derivatives can also be characterized in geometric terms. Specifically, Δ differs from δ_λ in that it removes the covariant part of the gauge transformation. Therefore, Δ acting on any covariant expression will give zero. For instance, if one constructs the field-strength, $F_{ij} \equiv \partial_i a_j - \partial_j a_i - i[a_i, a_j]$, one finds by explicit calculation

$$\Delta F_{ij} = 0. \quad (3.58)$$

It can also be checked that the covariant derivative, $D_i = \partial_i - i[a_i, \cdot]$ commutes with Δ ,

$$[\Delta, D_i] = 0. \quad (3.59)$$

In terms of Δ the equations (3.54) take the form

$$\begin{aligned} \Delta \Lambda^{(n)} &= M^{(n)} \\ \Delta A_i^{(n)} &= U_i^{(n)}. \end{aligned} \quad (3.60)$$

In the next section we will provide a method to solve these equations. Also note that since $\Delta^2 = 0$, it must be true that

$$\begin{aligned} \Delta M^{(n)} &= 0 \\ \Delta U_i^{(n)} &= 0. \end{aligned} \quad (3.61)$$

Indeed one should verify that (3.61) holds order by order. If (3.61) did not hold, this would signal an inconsistency in the SW map.

3.4 The Homotopy Operator

For simplicity, we begin by considering in detail the SW Map for the case of the gauge parameter Λ . Much of what we say actually applies to the other cases as well with minor modifications. In the previous section, we have seen that order by order in an expansion in θ , the SW map has the form:

$$\Delta \Lambda^{(n)} = M^{(n)}, \quad (3.62)$$

where $M^{(n)}$ depends only on $\Lambda^{(i)}$ with $i < n$. Clearly, if Δ could somehow be inverted, we could solve for $\Lambda^{(n)}$. But Δ is obviously not invertible, as $\Delta^2 = 0$. In particular, the

solutions of (3.62) are not unique, as if $\Lambda^{(n)}$ is a solution at order n in theta, then

$$\Lambda'^{(n)} = \Lambda^{(n)} + \Delta S^{(n)} \quad (3.63)$$

is also a solution for any $S^{(n)}(a_i, \partial_i a_j, \dots)$ of ghost number 0 and order n in theta⁴. That is, Δ acts like a coboundary operator in a cohomology theory, and the solutions that we are looking for are actually cohomology classes of solutions, unique only up to the addition of Δ -exact terms. Here, the ghost number plays a role analogous to that of the degree of a differential form (or (co)chain) in DeRham cohomology, and polynomials (of a_i , ∂_i , and v) of a given ghost number are the analogues of the chains themselves. Thus, we can think of the algebra of all such polynomials as the analogue of a (co)chain complex (e.g. compare to the grassmannian algebra of differential forms). Further, the consistency condition, $\Delta M^{(n)} = 0$ is just the statement the $M^{(n)}$ is necessarily a (co)cycle.

The formal existence of the SW Map is then equivalent to the statement that the cycle $M^{(n)}$ is actually Δ -exact for all n . Since we know that $\Delta^2 = 0$, this fact would follow as a corollary of the stronger statement that there is no non-trivial Δ cohomology in ghost number two (that there do not exist any Δ -closed, order n polynomials with ghost number two which are not also Δ -exact). To prove this stronger claim, we could proceed as follows. Suppose that we could construct an operator K (known as a chain homotopy operator⁵) acting on chains of ghost number two such that

$$K\Delta + \Delta K = 1. \quad (3.64)$$

Clearly, K must reduce ghost number by one, and therefore must be odd. Consider its action on a cycle M , (so $\Delta M = 0$)

$$(K\Delta + \Delta K)M = \Delta KM = M. \quad (3.65)$$

⁴These are precisely the ambiguities in the SW map that were first discussed in [71], where our operator Δ was called δ' .

⁵The name comes from homology theory: Suppose ∂ is a (co)boundary operator and g and h are (co)chain maps (maps between two chain complexes which commute with the boundary operator, $g\partial = \partial g$). If we can find a K s.t. $K\partial + \partial K = g - h$, then if α is a (co)cycle, $g(\alpha) - h(\alpha) = \partial K(\alpha)$, so g and h induce the same maps on homology. In simplicial homology, this holds if the chain maps are induced by homotopic maps between the underlying topological spaces. In our case, we take $g = 1$, $h = 0$, so the existence of K implies that any cycle α is exact, as $g(\alpha) = \alpha = \partial K(\alpha)$, and thus the cohomology is trivial.

Therefore, $M = \Delta\Lambda$, with $\Lambda = KM$, which not only shows that M is exact, but also computes explicitly a solution to the SW map. We note that this method of solution is nearly identical to the method used by Stora and Zumino [70] to solve the Wess-Zumino consistency conditions for non-Abelian anomalies. In fact, it was the parallels between these problems that motivated our current approach.

We now proceed to construct K . First we notice that $M^{(n)}$ depends on λ only through its derivative $\partial_i v$, as one can see by looking at the explicit expressions in the Appendix. The same is true for $U_i^{(n)}$ since it depends on λ only through Λ . It is convenient to define

$$b_i = \partial_i \lambda, \quad (3.66)$$

so that M and U_i can all be expressed as functions of a_i , b_i and their derivatives only. Further, as Δ commutes with *covariant* derivatives ($D_i = \partial_i - i[a_i, \cdot]$), it is convenient to rewrite $M^{(n)}$ by replacing all ordinary derivatives with covariant derivatives. After these replacements, we may naively attempt to consider $M^{(n)}$ an element of the noncommutative algebra generated freely by the a_i and b_i under the free action of the covariant derivatives D_i . The action of Δ on the generators of this algebra takes a very simple form (where $D^n = D_{i_1} \cdots D_{i_n}$),

$$\Delta(D^n)a_i = (D^n)b_i, \quad \Delta(D^m)b_i = 0 \quad [\Delta, D_i] = 0. \quad (3.67)$$

However, we note the action of the covariant derivatives on a_i and b_i is not actually free. In particular, if we define

$$F_{ij} = D_i a_j - D_j a_i + i[a_i, a_j], \quad (3.68)$$

and then consider,

$$\Delta F_{ij} = \Delta(D_i a_j - D_j a_i + i[a_i, a_j]) = D_i b_j - D_j b_i + i[b_i, b_j], \quad (3.69)$$

we see that the right hand side of this expression would be a non-vanishing element of the free algebra generated by the a_i , b_i , and D_i , while according to (3.58) it actually must vanish. The origin of these relations or constraints can be traced to the fact that the commutation

of ordinary partial derivatives,

$$\partial_i \partial_j - \partial_j \partial_i = 0, \quad \partial_i b_j - \partial_j b_i = 0, \quad (3.70)$$

is not manifest in the algebra, since we've defined $b_i = \partial_i v$ and rewritten the algebra in terms of covariant derivatives. Note that when written in terms of covariant derivatives, (3.70) becomes (3.69).

It will be much easier to construct the homotopy operator if we can solve the constraints while still using variables in which the action of Δ is still simple.⁶ Now, we claim that all the constraints that arise in this algebra arise as consequences of (3.70). First note that λ and a_i , under the action of ∂_i freely generate a noncommutative algebra with the only constraints being the commutation of the partial derivatives. Replacing $\partial_i \lambda$ with b_i is entirely analogous to replacing a gauge potential with its field strength. Of course, the only consequence of this replacement is that the field strength must be constrained to obey its Bianchi identity, which is precisely $\partial_i b_j - \partial_j b_i = 0$. Now, expressing elements of this algebra in terms of covariant derivatives is just a linear change of basis of generators, replacing the exterior derivation ∂_i with a combination of exterior and interior derivation $D_i = \partial_i - i[a_i, \cdot]$ and therefore cannot add any new constraints. In particular, we only have the two constraints acting on the original algebra, and can just translate them into constraints involving the D_i . With F_{ij} defined by (3.68), the constraints (3.70) just translate into,

$$(\partial_i \partial_j - \partial_j \partial_i)(\cdot) = [F_{ij}, \cdot] - i[D_i, D_j](\cdot) = 0, \quad (3.71)$$

$$\partial_i b_j - \partial_j b_i = \Delta F_{ij} = 0. \quad (3.72)$$

Note that both constraints can be written in terms of antisymmetric combinations of covariant derivatives and fields. This suggests that symmetric combinations of covariant derivatives and fields should be treated differently from antisymmetric combinations.

In particular, consider an algebra generated by $(D^p)F_{ij}$, treated here as independent

⁶In fact, the following construction has been made completely obsolete, as a much simpler method exists for obtaining the solution to all orders that only requires that a homotopy operator be defined to first order [76]. Nevertheless, we include the following for completeness.

variables, along with the totally symmetrized variables,

$$(D^n a)_s = \frac{1}{n!} \sum_{\sigma} D_{i_{\sigma(1)}} \cdots D_{i_{\sigma(n-1)}} a_{i_{\sigma(n)}}, \quad (3.73)$$

$$(D^n a)_s = \frac{1}{n!} \sum_{\sigma} D_{i_{\sigma(1)}} \cdots D_{i_{\sigma(n-1)}} a_{i_{\sigma(n)}}, \quad (3.74)$$

where the sum is over all permutations σ of $1, \dots, n$. Clearly, we can express any co-chain in terms of these variables by rewriting expressions in terms of symmetric and anti-symmetric parts (we make this more explicit at the end of this section, for pedagogical clarity). The question is to what extent such an expression is unique. Note that we only allow covariant derivatives to act on the field strengths. Since all the relations in our original algebra involved commutators of derivatives, and we do not allow freely acting covariant derivatives on the symmetrized variables, we do not expect any relations involving these totally symmetrized variables. However, we certainly do expect relations involving the freely acting covariant derivatives acting on the field strength, such as,

$$D_i F_{jk} + D_k F_{ij} + D_j F_{ki} = 0. \quad (3.75)$$

Formally, this means that the ideal of relations is generated by linear combinations of generators of the subalgebra $(D^p)F_{ij}$. As we will discuss presently, since all the operators we will define can be taken to commute with these freely acting covariant derivatives and field strengths, this will not matter.

First note that since Δ is a linear operator and commutes with all covariant derivatives, it annihilates the $(D^p)F_{ij}$ and acts very simply on the $(D^n a)_s$ and $(D^m b)_s$,

$$\begin{aligned} \Delta(D^n a)_s &= (D^n b)_s, \\ \Delta(D^m b)_s &= 0, \\ \Delta(D^p)F_{ij} &= 0. \end{aligned} \quad (3.76)$$

The last relation can also be interpreted as stating that Δ commutes with all the $(D^p)F_{ij}$, and therefore, the ideal of relations commutes with Δ as well. This means that we can proceed as if in a free algebra. In a free algebra, a homotopy operator is fairly easy to

construct. The primary difficulty is that since K is to invert an operator which acts like a graded derivation, it cannot itself obey the Leibniz rule. We can instead proceed by defining an infinitesimal form of the operator K , which does. In particular, to define K , we first define two operators ℓ and δ such that

$$\Delta\ell + \ell\Delta = \delta, \quad (3.77)$$

and then an operator T (a kind of integration operator) such that

$$T\delta M = M, \quad T(\ell M) = KM. \quad (3.78)$$

We think of the operator δ as an infinitesimal variation of $(D^n a)_s$ and $(D^n b)_s$ which can be integrated to the identity. It is also defined to annihilate the $(D^p)F_{ij}$,

$$\delta((D^p)F_{ij}) = 0. \quad (3.79)$$

The action of ℓ is defined by

$$\begin{aligned} \ell(D^n a)_s &= 0, \\ \ell(D^m b)_s &= (D^m a)_s, \\ \ell(D^p)_s F_{ij} &= 0. \end{aligned} \quad (3.80)$$

Finally, the integration operator T acting on any expression is implemented via the following procedure:

1. Choose the fields to be linearly dependent on t and δ to be the infinitesimal variation with respect to t ,

$$\begin{aligned} \delta(D^n a)_s &\rightarrow (D^n a)_s dt, \\ \delta(D^m b)_s &\rightarrow (D^m b)_s dt, \\ \delta(D^p)_s F_{ij} &\rightarrow 0, \\ (D^n a)_s &\rightarrow t(D^n a)_s, \\ (D^m b)_s &\rightarrow t(D^m b)_s, \\ (D^p)_s F_{ij} &\rightarrow (D^p)_s F_{ij}. \end{aligned} \quad (3.81)$$

That is, we transform any expression,

$$\begin{aligned} N((D^n a)_s, (D^m b)_s, \delta(D^n a)_s, \delta(D^m b)_s, (D^p)_s F_{ij}) \rightarrow \\ N(t(D^n a)_s, t(D^m b)_s, (D^n a)_s dt, (D^m b)_s dt, (D^p)_s F_{ij}). \end{aligned} \quad (3.82)$$

2. Integrate from $t = 0$ to $t = 1$. Thus,

$$\begin{aligned} TN((D^n a)_s, (D^m b)_s, \delta(D^n a)_s, \delta(D^m b)_s, (D^p)_s F_{ij}) = \\ \int_0^1 N(t(D^n a)_s, t(D^m b)_s, (D^n a)_s dt, (D^m b)_s dt, (D^p)_s F_{ij}). \end{aligned} \quad (3.83)$$

We now show by induction that these definitions do in fact yield a homotopy operator K . It is easy to see that $\Delta\ell + \ell\Delta = \delta$ holds when acting on $(D^n a)_s$ or $(D^m b)_s$ alone. Suppose then that the equation holds when acting on two monomials f and g of order less than or equal to r in $(D^n a)_s$ and $(D^m b)_s$. Then it follows that

$$(\Delta\ell + \ell\Delta)(fg) = ((\Delta\ell + \ell\Delta)f)g + f(\Delta\ell + \ell\Delta)g, \quad (3.84)$$

where all the cross terms have canceled out. By the induction hypothesis this expression is equal to $(\delta f)g + f\delta g$, which is just $\delta(fg)$. Thus $\Delta\ell + \ell\Delta = \delta$ holds on any monomial of degree greater than zero. Since this operator is distributive, (3.77) holds for any element of the algebra.

Finally, as promised, we show in more detail how we can actually write the co-chain M in the form suggested above. We begin with an expression for M as found by expanding the star product.

$$M^{(n)} = M^{(n)}(a, (\partial^k)_s a, (\partial^l)_s v), \quad (3.85)$$

where we choose to explicitly write the derivatives in symmetric form. By replacing $\partial(\cdot) \rightarrow D(\cdot) + i[a, \cdot]$, and $\partial v \rightarrow b$ the expression takes the form

$$M^{(n)} = M^{(n)}(a, b, (D^k)_s a, (D^l)_s b). \quad (3.86)$$

The difference $(D^k a)_s - D^k a$ contains terms that are proportional to the antisymmetric parts of DD or Da . But using the constraints we can make the following substitutions

$$[D_i, D_j](\cdot) \rightarrow -i[F_{ij}, \cdot], \quad D_i a_j - D_j a_i \rightarrow F_{ij} - i[a_i, a_j]. \quad (3.87)$$

This must be done recursively since the commutator term involving a 's above may again be acted on by D 's. But at each step, the number of possible D 's acting on a is reduced by one. After carrying out this procedure M will be written in terms of the desired variables.

3.5 Some Calculations

In this final section, we use the formalism we have developed to compute some low order terms of the SW map. The explicit form of the expansions of the Weyl-Moyal product used in this sections are included at the end of the section for completeness.

We focus mainly on solving for the gauge parameter Λ . At the zeroth order, if we expand $\delta_\lambda \Lambda = i\Lambda \star \Lambda$ we find

$$\delta_\lambda v = iv^2, \quad (3.88)$$

which is just the BRST transformation of v (3.45). At first order, we have

$$\Delta \Lambda^{(1)} = -\frac{1}{2} \theta^{ij} b_i b_j, \quad (3.89)$$

while at the second order we obtain

$$\Delta \Lambda^{(2)} = -\frac{i}{8} \theta^{ij} \theta^{kl} \partial_i b_k \partial_j b_l - \frac{1}{2} \theta^{ij} [b_i, \partial_j \Lambda^{(1)}] + i \Lambda^{(1)} \Lambda^{(1)}. \quad (3.90)$$

A solution of (3.89) has been found in [59] and is given by

$$\Lambda^{(1)} = \frac{1}{4} \theta^{ij} \{b_i, a_j\}. \quad (3.91)$$

We can reproduce this solution immediately by applying K to the expression $M^{(1)} = -\frac{1}{2} \theta^{ij} b_i b_j$. There are no problems at this level, since there are not enough derivatives for the constraints to show up. As explained in the previous section we proceed in two steps. We first apply ℓ

$$\ell(M^{(1)}) = -\frac{1}{2} \theta^{ij} (\delta a_i b_j - b_i \delta a_j), \quad (3.92)$$

then T to find

$$K(M^{(1)}) = \frac{1}{2} \theta^{ij} (b_i a_j + a_j b_i) \int_0^1 dt = \frac{1}{4} \theta^{ij} \{b_i, a_j\}. \quad (3.93)$$

The ambiguity in the first order solution as determined in [71] is proportional to

$$\tilde{\Lambda}^{(1)} = -2i\theta^{ij}[b_i, a_j] . \quad (3.94)$$

According to the previous discussion the ambiguity amounts to an exact cocycle, hence is of the form:

$$\tilde{\Lambda}^{(1)} = \Delta S^{(1)} , \quad (3.95)$$

where $S^{(1)}$ can be computed to be

$$S^{(1)} = K\tilde{\Lambda}^{(1)} = -i\theta^{ij}[a_i, a_j] . \quad (3.96)$$

Solutions at the second order have been found by various authors. In [63] the following solution is presented,⁷

$$\begin{aligned} \Lambda^{(2)} = & \frac{1}{32}\theta^{ij}\theta^{kl} \left(- \{b_i, \{a_k, i[a_j, a_l] + 4\partial_l a_j\}\} - i\{a_j, \{a_l, [b_i, a_k]\}\} \right. \\ & \left. + 2[[b_i, a_k] + i\partial_i b_k, \partial_j a_l] + 2i[[a_j, a_l], [b_i, a_k]] \right) , \end{aligned} \quad (3.97)$$

while in [72] the following solution is found,

$$\begin{aligned} \Lambda'^{(2)} = & \frac{1}{32}\theta^{ij}\theta^{kl} \left(- \{b_i, \{a_k, i[a_j, a_l] + 4\partial_l a_j\}\} - i\{a_j, \{a_l, [b_i, a_k]\}\} \right. \\ & \left. + 2[[b_k, a_i] + i\partial_i b_k, \partial_j a_l] \right) . \end{aligned} \quad (3.98)$$

According to our previous observation the difference between these two expressions must be of the form $\Delta S^{(2)}$. In fact, we find

$$\Lambda^{(2)} - \Lambda'^{(2)} = \frac{1}{16}\theta^{ij}\theta^{kl} \left([\partial_i a_k, [b_l, a_j] + [a_l, b_j]] - i[[a_k, b_i], [a_l, a_j]] \right) = \Delta S^{(2)} , \quad (3.99)$$

with $S^{(2)}$ given by

$$S^{(2)} = \frac{1}{16}\theta^{ij}\theta^{kl} [\partial_i a_k, [a_l, a_j]] . \quad (3.100)$$

This expression for $S^{(2)}$ can be obtained in the following way.

⁷The expression for $\Lambda^{(2)}$ found in [63] appears to contain a misprint. The correct formula (3.97) was in fact given to us personally by J. Wess.

Symmetrize $\Lambda^{(2)} - \Lambda'^{(2)}$ with respect to all derivatives and then use the substitution (3.87)

$$F_{ij} - D_i a_j - D_j a_i + i[a_i, a_j] = 0, \quad (3.101)$$

to introduce F rather than antisymmetric derivatives and fields.

$$\begin{aligned} \Lambda^{(2)} - \Lambda'^{(2)} &= \frac{1}{16} \theta^{ij} \theta^{kl} \left([F_{ik} + i[a_i, a_k], [b_l, a_j]] + i[[a_k, b_i], [a_j, a_l]] \right) \\ &= \frac{1}{16} \theta^{ij} \theta^{kl} [F_{ik}, [b_l, a_j]]. \end{aligned} \quad (3.102)$$

By applying K we immediately get

$$S^{(2)} = K(\Lambda^{(2)} - \Lambda'^{(2)}) = \frac{1}{32} \theta^{ij} \theta^{kl} [F_{ik}, [a_l, a_j]] \quad (3.103)$$

By substituting back the expression for F_{ik} and noting that

$$\theta^{ij} \theta^{kl} [[a_k, a_i], [a_j, a_l]] = 0 \quad (3.104)$$

we again recover (3.100). By following the same procedure we can compute directly a solution of (3.90) at the second order.

$$\begin{aligned} \Lambda''^{(2)} &= -\frac{1}{2} \theta^{ij} \left\{ a_i, \frac{1}{3} D_j \Lambda^{(1)} + \frac{i}{4} [a_j, \Lambda^{(1)}] \right\} + \theta^{ij} \theta^{kl} \left(-\frac{i}{16} [D_i a_k, D_j b_l] \right. \\ &\quad + [[a_i, a_k], \frac{1}{24} D_j b_l + \frac{i}{32} [a_j, b_l]] + \frac{1}{24} [D_i a_k, [a_j, b_l]] \\ &\quad + \frac{1}{8} \left(a_i \left(\frac{1}{3} D_j a_k - \frac{1}{3} D_k a_j + \frac{i}{2} [a_j, a_k] \right) b_l \right. \\ &\quad \left. - b_i \left(\frac{1}{3} D_j a_k - \frac{1}{3} D_k a_j + \frac{i}{2} [a_j, a_k] \right) a_l \right. \\ &\quad \left. + \left\{ \frac{1}{6} (D_i a_k - D_k a_i) + \frac{i}{4} [a_i, a_k], \{a_l, b_j\} \right\} \right) \right). \end{aligned} \quad (3.105)$$

As expected, the difference between our solution $\Lambda''^{(2)}$ (3.105) and the solution $\Lambda^{(2)}$ (3.97) is again of the form $\Delta S'^{(2)}$ (up to a term which vanishes by the constraint).

$$\begin{aligned} \Lambda''^{(2)} - \Lambda^{(2)} &= \theta^{ij} \theta^{kl} \left[\Delta \left(\frac{1}{24} ([a_j, [D_i a_k, a_l]] + 2(D_i a_k a_j a_l + a_l a_j D_i a_k) \right) \right. \\ &\quad \left. + \frac{1}{16} [a_i a_k, \Delta F_{jl}] \right]. \end{aligned} \quad (3.106)$$

A similar technique can be followed for the potential A_i . Moreover, if $\Lambda^{(n)}$ is changed by an amount $\Delta S^{(n)}$,

$$\Lambda^{(n)} \rightarrow \Lambda^{(n)} + \Delta S^{(n)}, \quad (3.107)$$

then the corresponding change in the potential is

$$A_i^{(n)} \rightarrow A_i^{(n)} + D_i S^{(n)}. \quad (3.108)$$

This follows from the fact that the equation of order n for the gauge field is always of the form

$$\Delta A_i^{(n)} = D_i \Lambda^{(n)} + \dots. \quad (3.109)$$

Notice that (3.108) is a consequence of the fact that the coboundary operator Δ commutes with the covariant derivative D_i .

3.5.1 Explicit expansions of Weyl-Moyal product

We collect here some useful expressions arising from the expansion of the Weyl-Moyal product. First, for simplicity, we will define

$$\partial_{I_n} = \partial_{i_1} \cdots \partial_{i_n} \quad (3.110)$$

$$\theta^{I_n J_n} = \theta^{i_1 j_1} \cdots \theta^{i_n j_n}. \quad (3.111)$$

We will expand out \star -products using Moyal's formula:

$$\begin{aligned} f(x) \star g(x) &= e^{\frac{i}{2} \theta^{ij} \partial_y \partial_z} f(y) g(z) |_{y=z=x} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} \right)^n \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \partial_{i_1} \cdots \partial_{i_n} f(x) \partial_{j_1} \cdots \partial_{j_n} g(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} \right)^n \theta^{I_n J_n} \partial_{I_n} f(x) \partial_{J_n} g(x). \end{aligned} \quad (3.112)$$

Inserting this expansion into (3.51) and requiring that the equation is satisfied order by order in θ , we find the following expression

$$\begin{aligned}
\delta_\lambda \Lambda^{(n)} - i \left\{ \Lambda^{(n)}, v \right\} &= \Delta \Lambda^{(n)} \\
&= \sum_{p=1}^{n-1} \left(\frac{i}{(n-p)!} \left(\frac{i}{2} \right)^{n-p} \theta^{I_{n-p} J_{n-p}} \{ \partial_{I_{n-p}} \Lambda_p, \partial_{J_{n-p}} v \} \right. \\
&\quad \left. + \frac{i}{(p-1)!} \left(\frac{i}{2} \right)^{p-1} \theta^{I_{p-1} J_{p-1}} \sum_{q=1}^{n-p} \partial_{I_{p-1}} \Lambda_q \partial_{J_{p-1}} \Lambda_{n-q-p+1} \right) \\
&\quad + \frac{i}{n!} \left(\frac{i}{2} \right)^n \theta^{I_n J_n} (\partial_{I_n} v) (\partial_{J_n} v)
\end{aligned} \tag{3.113}$$

Up to the second order this equation reads

$$0^{th} : \quad \delta_\lambda v = i v^2 \tag{3.114}$$

$$1^{st} : \quad \Delta \Lambda^{(1)} = -\frac{1}{2} \theta^{ij} b_i b_j \tag{3.115}$$

$$2^{nd} : \quad \Delta \Lambda^{(2)} = -\frac{i}{8} \theta^{ij} \theta^{kl} \partial_i b_k \partial_j b_l - \frac{1}{2} \theta^{ij} [b_i, \partial_j \Lambda^{(1)}] + i \Lambda^{(1)} \Lambda^{(1)} . \tag{3.116}$$

Analogously the equation (3.52) for the gauge potential A_i

$$\delta_\lambda A_i = \partial_i \Lambda - i [A_i \star \Lambda] \tag{3.117}$$

reads

$$0^{th} : \quad \Delta A_i^{(0)} = b_i \tag{3.118}$$

$$1^{st} : \quad \Delta A_i^{(1)} = D_i \Lambda^{(1)} - \frac{1}{2} \theta^{ij} \{ b_k, \partial_l a_i \} \tag{3.119}$$

$$\begin{aligned}
2^{nd} : \quad \Delta A_i^{(2)} &= D_i \Lambda^{(2)} + i [\Lambda^{(1)}, A_i^{(1)}] - \frac{1}{2} \theta^{kl} \{ b_k, \partial_l A_i^{(1)} \} \\
&\quad - \frac{1}{2} \theta^{kl} \{ \partial_k \Lambda^{(1)}, \partial_l a_i \} - \frac{i}{8} \theta^{kl} \theta^{mn} [\partial_k b_m, \partial_l \partial_n a_i] .
\end{aligned} \tag{3.120}$$

Chapter 4

Dipole Field Theories and Plane Wave Spacetimes

In the previous chapters, we have restricted ourselves to deformed geometries associated with spaces of Euclidean signature. In this chapter and the next, we move to interesting modifications of geometry which have to do with the behavior of string theory in backgrounds with non-trivial Lorentzian metrics. We begin, in this chapter, with an example which is uniquely interesting because of its simplicity, and is the result of work done in collaboration with Ori Ganor [16]. We will focus on the D-brane probes of certain particularly tractable plane-wave spacetimes. We argue that the effective field theory on D3-branes in a plane-wave background with 3-form flux is a nonlocal deformation of Yang-Mills theory. In the case of NSNS flux, it is a dipole field theory with lightlike dipole vectors. For an RR 3-form flux the dipole theory is strongly coupled. We propose a weakly coupled S-dual description for it. The S-dual description is local at any finite order in string perturbation theory but becomes nonlocal when all perturbation theory orders are summed together.

4.1 Introduction

The restrictions imposed by the conditions of global Lorentz invariance and locality play central roles in our understanding of the formal properties of quantum field theories. However, as we have seen in the last chapter, neither of these conditions appears to be fundamental in string theory. Thus, it is interesting to consider simple situations where they are relaxed. In particular, we will examine the properties of D-branes in certain plane-wave backgrounds with strong 3-form fields. As we will show in detail, the low energy effective theory describing the fluctuations of these D-branes is a non-local, Lorentz violating dipole theory [77]-[79].

Typical interaction terms in the Lagrangian of this field theory are of the form,

$$\int \phi_1(\vec{x}) \phi_2(\vec{x} + \vec{L}_1) \phi_3(x + \vec{L}_1 + \vec{L}_2) \cdots d^4 \vec{x}, \quad (4.1)$$

where ϕ_i are fields and the \vec{L}_i are fixed world-volume vectors. Roughly speaking, the non-locally coupled fields ϕ_i correspond to stretched open strings with end-points that are separated by \vec{L}_i and with angular momentum along planes transverse to the brane. These strings are stabilized by the presence of strong 3-form fluxes with legs aligned along the dipole vectors as well as the plane of rotation [78].

An exciting application of string theory with strong 3-form field strengths is the AdS_3 / CFT_2 correspondence [80]. Unfortunately, progress had been limited by the fact that string theory in AdS backgrounds with RR field strengths are difficult to analyze exactly. However, the authors of [81] have shown that a particularly tractable limit of the AdS/CFT correspondence can be obtained by taking the Penrose limit of type-IIB string theory on $AdS_5 \times S^5$ to obtain a plane-wave background. They were able to precisely match the properties of a certain subsector of $\mathcal{N} = 4$ Super-Yang-Mills CFT (operators with large R-charge) with the exact results of [82]-[84] for strings in plane-wave backgrounds.

Similarly, one can consider the Penrose limits of $AdS_3 \times S^3 \times T^4$ [85, 86]. As IIB has two three-form field strengths, H^1 (NSNS) and H^2 (RR), one finds a pair of models which

are related by S-duality. The Penrose limit of the theory with H^1 flux is

$$ds^2 = dx^+ dx^- + \mu x^i x^i (dx^+)^2 - dx^a dx^a - dx^i dx^i, \quad (4.2)$$

$$H^1 = -\mu dx^+ \wedge (dx^6 \wedge dx^7 + dx^8 \wedge dx^9), \quad (4.3)$$

$$e^\phi = g_s, \quad (4.4)$$

where ds^2 is the interval in string frame, $x^\pm = x^0 \pm x^1$, the x^a are coordinates on T^4 with $a = 2, \dots, 5$ and $i = 6, \dots, 9$. The Penrose limit of the S-dual configuration is

$$ds^2 = dx^+ dx^- + \mu x^i x^i (dx^+)^2 - dx^a dx^a - dx^i dx^i, \quad (4.5)$$

$$H^2 = \mu dx^+ \wedge (dx^6 \wedge dx^7 + dx^8 \wedge dx^9), \quad (4.6)$$

$$e^\phi = \frac{1}{g_s}. \quad (4.7)$$

Exact results for the spectrum of both models were obtained in [85, 86]. Further, open strings and D-branes in these and other plane-wave backgrounds have been studied in [87]-[99].

In this chapter we will study the interactions of the low energy effective theory of the D-brane excitations. We will show that N D3-brane probes of the plane-wave background (4.2)-(4.4) are exactly described at low energies by a nonlocal $U(N)$ dipole gauge theory [77] with a lightlike dipole vector \vec{L} proportional to μ .

A more complicated problem is the description of N D3-brane probes of the pp-wave background (4.5)-(4.7), which has RR flux. It is related to the S-dual description of the lightlike dipole theory. We attack this problem by first studying the S-dual description of a $U(1)$ lightlike dipole theory and then guessing the generalization of that result to a $U(N)$ gauge group. We find that in any finite order of string perturbation theory the interactions of the D3-brane probes of the pp-wave background (4.5)-(4.7) are local. Yet our result suggests that summing the local interactions to all orders in perturbation theory exhibits an intrinsic nonlocality with a characteristic length proportional to the string coupling constant, g_s .

The chapter is organized as follows. In section 4.2 we review the definition and salient features of dipole theories. In section 4.3 we identify the lightlike dipole theory as the low

energy description of D3-branes in the pp-wave background (4.2)-(4.4). In section 4.4 we analyze the S-dual of the $U(1)$ lightlike dipole theories and conjecture an extension of the result to $U(N)$, We conclude in section 4.5 with a list of possible extensions of our work.

4.2 Definition of Dipole Theories

The dipole field theories that we will work with in this chapter are nonlocal field theories that are deformations of $\mathcal{N} = 4$ SYM. The Lagrangian of $\mathcal{N} = 4$ SYM is

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=4} &= \frac{1}{g^2} \text{tr} \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \sum_{I=1}^6 D_\mu \Phi^I D^\mu \Phi^I + i \sum_{a=1}^4 \bar{\psi}_a^{\dot{\alpha}} \sigma^{\mu\alpha}_{\dot{\alpha}} D_\mu \psi_\alpha^a \right\} \\ &+ \frac{1}{g^2} \text{tr} \left\{ \sum_{I < J} [\Phi^I, \Phi^J]^2 + \epsilon^{\alpha\beta} \sum_{I,a,b} \gamma_{ab}^I \Phi^I \psi_\alpha^a \psi_\beta^b + \epsilon_{\dot{\alpha}\dot{\beta}} \sum_{I,a,b} \gamma^{Iab} \Phi^I \bar{\psi}_a^{\dot{\alpha}} \bar{\psi}_b^{\dot{\beta}} \right\}, \\ D_\mu \Phi^I &\equiv \partial_\mu \Phi^I + i[A_\mu, \Phi^I].\end{aligned}\tag{4.8}$$

Here Φ^I ($I = 1 \dots 6$) are adjoint scalar fields of $U(N)$ which transform as a vector of the R-symmetry group $Spin(6)$. The ψ_α^a ($a = 1 \dots 4$) are adjoint Weyl fermions in the $\mathbf{4}$ of $Spin(6)$. Their complex conjugate fields $\bar{\psi}_a^{\dot{\alpha}}$ transform in the complex conjugate representation $\bar{\mathbf{4}}$ of $Spin(6)$. γ^{Iab} are the Clebsch-Gordan coefficients of $Spin(6)$ and $\sigma^{\mu\alpha}_{\dot{\alpha}}$ are Pauli matrices.

The dipole theories are obtained from $\mathcal{N} = 4$ SYM by the following steps (see [79] for more details):

1. Define the complex linear combinations of the 6 scalar fields of (4.8):

$$Z_k \equiv \Phi_{2k-1} + i\Phi_{2k}, \quad \bar{Z}_k \equiv \Phi_{2k-1} - i\Phi_{2k}, \quad k = 1, 2, 3,$$

and assign a constant space-time 4-vector \vec{L}_k to each scalar field Z_k .

2. Modify the covariant derivatives of the scalar fields so that $D_\mu Z_k$ at the space-time point x will be:

$$D_\mu Z_k(x) \equiv \partial_\mu Z_k(x) - iA_\mu(x) Z_k(x) + iZ_k(x) A_\mu(x) + \frac{1}{2} \vec{L}_k Z_k(x). \tag{4.9}$$

Note that the fields Z_k are $N \times N$ matrices in the adjoint representation of $U(N)$. Thus, equation (4.9) implies that the quanta of the fields Z_k are dipoles whose ends are at $x \pm \frac{1}{2}\vec{L}_k$. The gauge transformation of the scalar fields is

$$Z_k(x) \mapsto \Omega^{-1}(x - \frac{1}{2}\vec{L}_k) Z_k(x) \Omega(x + \frac{1}{2}\vec{L}_k),$$

where $\Omega(x) \in U(N)$ is the gauge group element.

3. In order to preserve $U(N)$ gauge invariance we have to modify the definition of the commutators in (4.8) to:

$$[Z_k, Z_l]_{(x)} \rightarrow Z_k(x - \frac{1}{2}\vec{L}_l) Z_l(x + \frac{1}{2}\vec{L}_k) - Z_l(x - \frac{1}{2}\vec{L}_k) Z_k(x + \frac{1}{2}\vec{L}_l).$$

4. We also need to modify the interactions of the fermions with the scalars so as to be gauge invariant. This can be done by assigning to the fermions their own dipole-vectors. To find the appropriate assignment we need to correlate the dipole-vector of the various fields with their $Spin(6) = SU(4)$ R-symmetry charges, as follows. The parameters \vec{L}_k that define the dipole theory can be combined into a single linear map $\Upsilon : su(4) \rightarrow \mathbb{R}^{3,1}$ from the Lie algebra of the R-symmetry group to a spacetime 4-vector. Using the inner product on $su(4)$, Υ can be represented as an $su(4)$ -valued spacetime 4-vector. In the representation **6** of $su(4)$ we can take Υ to be

$$\Upsilon \xrightarrow{\mathbf{6}} \begin{pmatrix} 0 & \vec{L}_1 & 0 & 0 & 0 & 0 \\ -\vec{L}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \vec{L}_2 & 0 & 0 \\ 0 & 0 & -\vec{L}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vec{L}_3 \\ 0 & 0 & 0 & 0 & -\vec{L}_3 & 0 \end{pmatrix}. \quad (4.10)$$

Now we can define the interactions of the fermions. We need to write Υ in the representation **4** of $su(4)$ and find a basis of this representation where Υ is diagonal.

It will then have the following form:

$$\Upsilon \xrightarrow{\mathbf{4}} \begin{pmatrix} \vec{\lambda}_1 & 0 & 0 & 0 \\ 0 & \vec{\lambda}_2 & 0 & 0 \\ 0 & 0 & \vec{\lambda}_3 & 0 \\ 0 & 0 & 0 & \vec{\lambda}_4 \end{pmatrix},$$

with the definitions

$$\begin{aligned} \vec{\lambda}_1 &= \frac{1}{2}(\vec{L}_1 + \vec{L}_2 + \vec{L}_3), \\ \vec{\lambda}_2 &= \frac{1}{2}(\vec{L}_1 - \vec{L}_2 - \vec{L}_3), \\ \vec{\lambda}_3 &= \frac{1}{2}(-\vec{L}_1 + \vec{L}_2 - \vec{L}_3), \\ \vec{\lambda}_4 &= \frac{1}{2}(-\vec{L}_1 - \vec{L}_2 + \vec{L}_3). \end{aligned} \tag{4.11}$$

The Weyl fermions ψ_α^a ($a = 1 \dots 4$) of (4.8), which are in the $\mathbf{4}$ of $su(4)$, should be assigned the dipole vectors $\vec{\lambda}^a$ and their complex conjugate fields should be assigned $(-\vec{\lambda}^a)$. To get a gauge invariant Lagrangian we need to replace all the commutators of a scalar and a fermion with:

$$[Z_k, \psi^a]_{(x)} \rightarrow Z_k(x - \frac{1}{2}\vec{\lambda}_a)\psi(x + \frac{1}{2}\vec{L}_k) - \psi(x - \frac{1}{2}\vec{L}_k)Z_k(x + \frac{1}{2}\vec{\lambda}_a).$$

5. Since the gauge bosons have vanishing dipole vectors, preserving any supersymmetry requires that some of the fermions have vanishing dipole vectors [79]. In particular, to preserve $\mathcal{N} = 2$ we may choose $\vec{\lambda}_1 = -\vec{\lambda}_2 = \vec{L}_2 = \vec{L}_3 = \vec{L}$ and $\vec{\lambda}_3 = \vec{\lambda}_4 = \vec{L}_1 = 0$.

These rules can be recast as a redefinition of the product of two fields. The modified product of any two fields $\Xi_1(x), \Xi_2(x)$ (scalar, fermionic or gauge) is defined in a way somewhat reminiscent of noncommutative geometry [100, 101]:

$$(\Xi_1 * \Xi_2)_{(x)} \equiv e^{\frac{i}{2}\langle \Upsilon^\mu, \hat{R}_1 \rangle \frac{\partial}{\partial z_\mu} - \frac{i}{2}\langle \Upsilon^\mu, \hat{R}_2 \rangle \frac{\partial}{\partial y_\mu}} (\Xi_1(y)\Xi_2(z))|_{y=z=x}, \tag{4.12}$$

where \hat{R}_i ($i = 1, 2$) is the ($su(4)$ -valued) R-symmetry charge operator acting on Ξ_i and $\langle \cdot, \cdot \rangle$ is the Killing form on $su(4)$ (see [79] for more details).

Special cases of dipole theories have been discussed in [102, 103] and various aspects of the theories have been explored in [104]-[109].

Lightlike dipole-vectors

Define the linear vector space $W \subset \mathbb{R}^{3,1}$ to be the image of the map $\Upsilon : su(4) \rightarrow \mathbb{R}^{3,1}$ defined in (4.10). In terms of the fundamental dipole vectors that were introduced in (4.10):

$$W = \text{Span}\{\vec{L}_1, \vec{L}_2, \vec{L}_3\}.$$

We will define the dipole theory to be *lightlike* if W is 1-dimensional and null, i.e.

$$\vec{L}_i \cdot \vec{L}_j = 0, \quad i, j = 1, 2, 3.$$

As we shall see in section 4.4, lightlike dipole theories are easier to analyze than the generic dipole theories. This is similar to Yang-Mills theory on a noncommutative space that simplifies when the noncommutativity parameter is lightlike [110]. Lightlike deformation parameters have also been used in the context of the noncommutative (2, 0)-theory [111]-[113].

4.3 Lightlike Dipole Theories and NSNS Plane-Waves

In this section we will show that the low energy effective actions describing appropriately oriented D3-branes in a plane-wave background with a strong lightlike NSNS 3-form flux are lightlike dipole theories. The orientation of the D3-branes must be such that, in the notation of (4.2)-(4.4), the $+$, $-$ directions are longitudinal and the x^i ($i = 6 \dots 9$) directions are transverse.

4.3.1 Geometric engineering of dipole-theories

To obtain a lightlike dipole theory we consider a background in which probe D3 branes have a small timelike dipole vector and then we perform a large boost. For simplicity,

assume that all the dipole vectors which are encoded in Υ are in the x^1 direction. In this case Υ reduces to a single element in the Lie algebra $su(4)$ which, in the representation **6**, we can write as a 6×6 antisymmetric matrix $2\pi\alpha'\hat{Q}$.

As was shown in [79], a $U(N)$ dipole theory with dipole vectors along x^1 described by $2\pi\alpha'\hat{Q}$ arises as the low-energy effective action of N D3-brane probes in the string theory background,

$$\begin{aligned} ds^2 &= dt^2 - \frac{1}{1 + \vec{x}^\top \hat{Q}^\top \hat{Q} \vec{x}} (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - d\vec{x}^\top d\vec{x} + \frac{(d\vec{x}^\top \hat{Q} \vec{x})^2}{1 + \vec{x}^\top \hat{Q}^\top \hat{Q} \vec{x}} \\ B &= \frac{1}{2} \frac{d\vec{x}^\top \hat{Q} \vec{x}}{1 + \vec{x}^\top \hat{Q}^\top \hat{Q} \vec{x}} \wedge dx^1, \quad e^{2(\phi - \phi_0)} = \frac{1}{1 + \vec{x}^\top \hat{Q}^\top \hat{Q} \vec{x}}, \end{aligned}$$

where $\vec{x} = (x^4, \dots, x^9)$. We can obtain a theory with a lightlike dipole vector by infinitely boosting this background along x^1 . As the dipole vector prior to the boost has a magnitude set by $2\pi\alpha'\hat{Q}$, we must simultaneously scale $\hat{Q} \rightarrow 0$ to obtain a lightlike dipole vector which has finite components in this limit. Thus, let

$$x^1 = \gamma(x^{1'} + vt'), \quad t = \gamma(t' + vx^{1'}), \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2}}$$

and take $v \rightarrow 1$ while keeping

$$\gamma\hat{Q} \equiv Q = \text{finite}.$$

Defining $x^\pm \equiv t' \pm x^{1'}$ we find the background

$$\begin{aligned} ds^2 &= dx^+ dx^- + (\vec{x}^\top Q^\top Q \vec{x})(dx^+)^2 - (dx^2)^2 - (dx^3)^2 - d\vec{x}^\top d\vec{x} \\ B &= \frac{1}{2} d\vec{x}^\top Q \vec{x} \wedge dx^+, \quad e^\phi = g_s. \end{aligned} \tag{4.13}$$

In order to preserve $\mathcal{N} = 2$ supersymmetry, we take

$$2\pi\alpha'Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L^- & 0 & 0 \\ 0 & 0 & -L^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L^- \\ 0 & 0 & 0 & 0 & -L^- & 0 \end{pmatrix}. \tag{4.14}$$

Concretely, we note that the dipole vectors for the fields in this background are of the form $\vec{L} = \pm(L^-, -L^-, 0, 0)$. If we define

$$\mu \equiv \frac{L^-}{2\pi\alpha'}, \quad (4.15)$$

we see that this background (4.13) is exactly the NSNS plane-wave of equation (4.2)-(4.4),

$$\begin{aligned} ds^2 &= dx^+ dx^- + \mu x^i x^i (dx^+)^2 - dx^a dx^a - dx^i dx^i, \\ H^1 &= -\mu dx^+ \wedge (dx^6 \wedge dx^7 + dx^8 \wedge dx^9), \\ e^\phi &= g_s, \end{aligned} \quad (4.16)$$

where again $a = 2, \dots, 5$ and $i = 6, \dots, 9$.

Note that L^- , the characteristic length scale of nonlocality, can be made arbitrarily big by a coordinate transformation that rescales x^+ . It is therefore obvious that the excited open string states decouple from the low energy lightlike dipole theory. Furthermore, since the lightlike dipole theory is a limit of a dipole theory with spacelike dipole vectors and since the latter can be constructed as a certain limit of compactified noncommutative $\mathcal{N} = 4$ Super Yang-Mills theory [77], it follows that the lightlike dipole theory is unitary.

4.3.2 Lightcone string theory in the NSNS background

Using the exact results of [81, 84] (extended by [97] to the open string case) for string theory in the NSNS plane-wave background (4.16), we will show directly that the open string interactions are modified by the phases one would expect for a lightlike dipole deformation.

In order to facilitate future comparisons to the RR case, we consider the plane wave background in the GS formalism. First, we define the complex worldsheet scalar fields

$$Z_1 \equiv X_6 + iX_7, \quad Z_2 \equiv X_8 + iX_9.$$

In order to simplify the analysis of the interactions in lightcone gauge, it is conventional to fix $X^+ = p^+ \tau$ and additionally require that the string length be $\ell = 2\pi\alpha' p^+$. Just as in [81, 84], we will find it useful to split our fermions into positive and negative chirality fermions with respect to Γ^{6789} . We use S to denote the positive chirality fermions. As the

negative chirality fermions and the scalars X^a , $a = 2, \dots, 5$ remain free and massless, we will ignore them. The resulting action in lightcone gauge is then given by,

$$S = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi\alpha'p^+} d\sigma \left[\frac{1}{2} \sum_{k=1}^2 \left(|\dot{Z}_k|^2 - |Z'_k + i\mu Z_k|^2 \right) + i\bar{S} (\sigma^0 \partial_0 + \sigma^1 (\partial_1 - \mu\Gamma^{67})) S \right] \quad (4.17)$$

There exists a field redefinition that, locally in σ , transforms this action into that of a free string. This transformation is [84, 97]

$$\tilde{Z}_k(\sigma) \equiv e^{i\mu\sigma} Z_k(\sigma), \quad \tilde{S}(\sigma) \equiv e^{-\mu\Gamma^{67}\sigma} S(\sigma). \quad (4.18)$$

In terms of the new fields the action is simply

$$S = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi\alpha'p^+} d\sigma \left[\frac{1}{2} \sum_{k=1}^2 \left(|\dot{\tilde{Z}}_k|^2 - |\tilde{Z}'_k|^2 \right) + i\tilde{S} (\sigma^0 \partial_0 + \sigma^1 \partial_1) \tilde{S} \right] \quad (4.19)$$

Note that the transformation (4.18) can change the boundary conditions of various fields. For closed strings, the transformed fields no longer satisfy periodic boundary conditions and the closed string spectrum in the plane-wave background differs from that of the free string. However, the spectrum of open strings with Dirichlet boundary conditions is unaltered. Instead, the interactions are modified in an interesting way as we will discuss presently.

4.3.3 Lightlike dipole-theories on D-branes in a plane-wave background

Consider a D1-brane that is extended in the x^+ , x^- directions. The extension of the discussion to D3-branes is straightforward. The open string excitations are described in lightcone gauge by the action (4.17) with the boundary conditions

$$Z_k(0) = Z_k(2\pi\alpha'p^+) = 0, \quad 0 = S_L(0) - S_R(0) = S_L(2\pi\alpha'p^+) - S_R(2\pi\alpha'p^+).$$

As the transformation (4.18) does not affect these boundary conditions, the spectrum of Dirichlet-Dirichlet open strings ending on a D1-brane in this NSNS plane-wave background is the same as the flat space spectrum. The interactions, however, receive extra phases that precisely reproduce the interactions described in section 4.2. Consider, for example, a

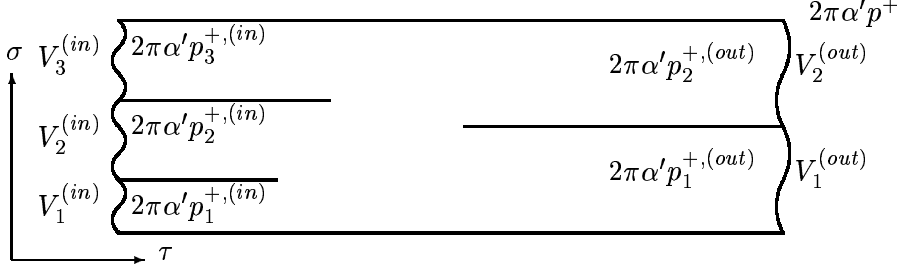


Figure 4.1: Scattering amplitude in the lightcone formalism.

tree level diagram that describes the scattering of open string states with vertex operators $V_1^{(in)}, \dots, V_{n_i}^{(in)}$ into open string states with vertex operators $V_1^{(out)}, \dots, V_{n_f}^{(out)}$ (see Figure 4.3.3). When written in terms of \tilde{Z}_k and \tilde{S} , these vertex operators should have the same form as the usual free Dirichlet-Dirichlet open string vertex operators. In fact, one might naively guess that as $\tilde{Z}_k = e^{i\mu\sigma} Z_k$ the relation should be

$$V_j^{(in)}(Z_k(\sigma), \dots) = \tilde{V}_j^{(in)}(e^{-i\mu\sigma} Z_k(\sigma), \dots), \quad (4.20)$$

where $\tilde{V}_j^{(in)}$ is the free string vertex operator that corresponds to the free string state with the same labels. This, of course, would give us the same amplitudes as in the free string case. However, note that if we let $p_j^{+, (in)}$ be the lightcone momentum of the j^{th} incoming string state, the parameter σ for that state is in the range

$$2\pi\alpha' \sum_{k=1}^{j-1} p_k^{+, (in)} \leq \sigma \leq 2\pi\alpha' \sum_{k=1}^j p_k^{+, (in)},$$

which means that the prescription (4.20) for defining the vertex operator contains phase factors which depend on the position of the insertion of the operator along the string. This cannot be correct.

We can solve this problem by replacing σ with $\sigma' = \sigma - 2\pi\alpha' \sum_{k=1}^{j-1} p_k^{+, (in)}$ (which is the

distance from the beginning of the j^{th} string) so

$$0 \leq \sigma' \leq 2\pi\alpha' p_j^{+, (in)} \quad (4.21)$$

$$V_j^{(in)}(Z_k(\sigma), \dots) = \tilde{V}_j^{(in)}(e^{-i\mu\sigma'} Z_k(\sigma), \dots). \quad (4.22)$$

This modification leads to overall phase shifts in the vertex operators as compared to the theory in flat space. To calculate them, we just need to know the Z_k and S dependence of the vertex operators. More formally, on the worldsheet there is a global $U(1)$ symmetry which acts on the Z_k by $Z_k \rightarrow e^{i\theta} Z_k$ (and analogously on the fermions, which we neglect for simplicity). A general vertex operator will transform under this $U(1)$ as $V^{(j)} \rightarrow e^{iq^{(j)}\theta} V^{(j)}$. Noting that $\mu = \frac{L^-}{2\pi\alpha'}$, it is easy to see that the definitions (4.22) and (4.20) differ by the phase,

$$\exp \left\{ i \sum_{l=1}^{j-1} q^{(j)} L^- p_l^{+, (in)} \right\}.$$

If we let $p_r^{+, (out)}$ be the lightcone momentum of the r^{th} outgoing string state, momentum conservation requires $p^+ = \sum_{j=1}^{n_i} p_j^{+, (in)} = \sum_{r=1}^{n_f} p_r^{+, (in)}$ and we see that the overall phase for the entire amplitude is

$$\exp \left\{ i \sum_{j=1}^{n_i} \sum_{l=1}^{j-1} q^{j, (in)} L^- p_l^{+, (in)} - i \sum_{r=1}^{n_f} \sum_{s=1}^{r-1} q^{r, (out)} L^- p_s^{+, (out)} \right\}.$$

It is not hard to see that this is exactly the same phase as the one we get by Fourier expanding the Super Yang-Mills action of a D-brane and replacing every product with the modified $*$ -product (4.12).

4.4 Proposal for the S-dual Theory

In this section we will present our proposal for the S-dual of the lightlike dipole theories. We will begin with an analysis of a dipole theory with a $U(1)$ gauge group and a single fermion (known as dipole QED [105, 108]) and then proceed to present our conjecture about a dipole theory with an $SU(N)$ or $U(N)$ gauge group.

The field contents of $U(1)$ dipole QED (without any supersymmetry) is:

A_μ the $U(1)$ gauge field,
 ψ a Dirac fermion with dipole vector \vec{L} .

The Lagrangian is

$$\mathcal{L} = \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2g^2} \bar{\psi} \gamma^\mu D_\mu \psi, \quad D_\mu \psi \equiv \partial_\mu \psi - i[A_\mu(x + \frac{1}{2}\vec{L}) - A_\mu(x - \frac{1}{2}\vec{L})]\psi. \quad (4.23)$$

Here \vec{L} is the constant dipole-vector and we assume that it is spacelike or null.

As shown in [105, 108], the Feynman rules of this theory are identical to those of ordinary QED, with the following modification of the interaction vertex,

$$ig\gamma^\mu \rightarrow ig\gamma^\mu \times 2i \sin \frac{p \cdot L}{2}, \quad (4.24)$$

where p is the outgoing momentum of the photon. In particular, this means that the photon self-energy at one-loop just gets an extra factor of

$$2i \sin \frac{p \cdot L}{2} \times 2i \sin \frac{-p \cdot L}{2} = 4 \sin^2 \frac{p \cdot L}{2} \quad (4.25)$$

as compared to the QED result. This suggests that the $U(1)$ theory is IR free, just like ordinary QED. Thus, our application of S-duality in the $U(1)$ case will be somewhat formal, and should be considered simply as a motivation for the conjecture in the $U(N)$ case.

To find the S-dual description we will adopt the standard method of using a Lagrange multiplier for the field strength.¹ We treat $F_{\mu\nu}$ as an independent field subject to the Bianchi identity $\epsilon^{\mu\nu\tau\rho} \partial_\nu F_{\tau\rho} = 0$ which we implement with a Lagrange multiplier. Of course, this method requires that the gauge field A_μ does not appear explicitly in the Lagrangian. Unlike in ordinary QED, here we can eliminate A_μ by performing a redefinition of variables [77]

$$\psi^{(inv)}(x) \equiv e^{-\frac{i}{2} \int_{-1}^1 L^\mu A_\mu(x + \frac{s}{2}\vec{L}) ds} \psi(x), \quad (4.26)$$

¹A similar method was used in [114] to study S-duality for Super Yang-Mills theory on a noncommutative $R^{3,1}$.

so that $\psi^{(inv)}$ is a $U(1)$ -neutral field. This is the analog of the Seiberg-Witten map [59] for dipole theories. Just as in that case, this transformation results in a theory with ordinary gauge symmetry perturbed by an infinite number of irrelevant interactions. In particular, since

$$D_\mu \psi(x) = e^{-\frac{i}{2} \int_{-1}^1 L^\nu A_\nu(x + \frac{s}{2} \vec{L}) ds} \left[\partial_\mu \psi^{(inv)}(x) + \frac{i}{2} \psi^{(inv)}(x) \int_{-1}^1 L^\nu F_{\mu\nu}(x + \frac{s}{2} \vec{L}) ds \right],$$

we can define

$$D^F_\mu \psi^{(inv)}(x) \equiv \partial_\mu \psi^{(inv)}(x) + \frac{i}{2} \psi^{(inv)}(x) \int_{-1}^1 L^\nu F_{\mu\nu}(x + \frac{s}{2} \vec{L}) ds,$$

to get the Lagrangian

$$\mathcal{L}_1 = \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2g^2} \bar{\psi}^{(inv)} \gamma^\mu D^F_\mu \psi^{(inv)}. \quad (4.27)$$

Thus, as promised, the explicit dependence on A_μ has been removed. Further, note that making the replacements $\psi \rightarrow g\psi$, and $A_\mu \rightarrow gA_\mu$ in the above Lagrangian and writing

$$\int_{-1}^1 ds F_{\mu\nu}(x + \frac{s}{2} \vec{L}) = \frac{2 \sin \frac{i}{2} L \cdot \partial}{\frac{i}{2} L \cdot \partial} F_{\mu\nu}(x), \quad (4.28)$$

we see that

$$\mathcal{L}_1 = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \bar{\psi}^{(inv)} \gamma^\mu \left(\partial_\mu + ig L^\nu \frac{\sin \frac{i}{2} L \cdot \partial}{\frac{i}{2} L \cdot \partial} F_{\mu\nu}(x) \right) \psi^{(inv)}$$

is just a free theory perturbed by an infinite number of higher derivative interactions with couplings of the form $gL \times L^{2n}$.

We can now easily find the S-dual theory by treating $F_{\mu\nu}$ as an independent variable and adding a Lagrange multiplier to the Lagrangian (4.27),

$$\mathcal{L}_2 = \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2g^2} \bar{\psi}^{(inv)} \gamma^\mu D^F_\mu \psi^{(inv)} + \frac{1}{8\pi} \tilde{A}_\mu \epsilon^{\mu\nu\tau\rho} \partial_\nu F_{\tau\rho}.$$

Except for the kinetic term, $F_{\mu\nu}$ appears linearly in \mathcal{L}_2 . Thus, we can integrate it out to get,

$$\mathcal{L}' = \frac{1}{4g'^2} \left(\tilde{F}_{\mu\nu} - \frac{1}{\pi g'^2} \epsilon_{\mu\nu\tau\rho} L^\tau \int_{-1}^1 \tilde{J}^\rho(x + \frac{s}{2} \vec{L}) ds \right)^2 + \frac{1}{2g'^2} \bar{\psi} \gamma^\mu \partial_\mu \psi. \quad (4.29)$$

where we have defined

$$g' = \frac{4\pi}{g}, \quad \tilde{\psi} \equiv \frac{4\pi}{g^2}\psi, \quad \tilde{J}_\mu \equiv i\tilde{\psi}\gamma_\mu\tilde{\psi}. \quad (4.30)$$

If we make a further redefinition of the fields,

$$\hat{\psi} \equiv \frac{1}{g'}\tilde{\psi}, \quad \hat{F} \equiv \frac{1}{g'}\tilde{F}, \quad \hat{L}^\tau \equiv \frac{1}{g'^2}L^\tau, \quad (4.31)$$

we see that (4.29) can be rewritten as

$$\mathcal{L}' = \frac{1}{4} \left(\hat{F}_{\mu\nu} - \frac{g'\hat{L}^\tau}{\pi} \epsilon_{\mu\nu\tau\rho} \int_{-1}^1 \hat{J}^\rho(x + \frac{s}{2}g'^2\hat{L}) ds \right)^2 + \frac{1}{2} \hat{\psi} \gamma^\mu \partial_\mu \hat{\psi}. \quad (4.32)$$

If we add minimally coupled scalars to the QED Lagrangian (4.23), with the same dipole vector \vec{L} , the expression of the S-dual \mathcal{L}' becomes more complicated because the interactions are quadratic in the gauge field. The dual Lagrangian simplifies for a lightlike dipole vector. To see this, we will fix the QED lightcone gauge $A_- = 0$. In this gauge the redefinition (4.26) becomes simply $\psi^{(inv)}(x) \equiv \psi(x)$. Following the same steps that led to (4.29) with minimally coupled scalars added we find that the dual Lagrangian can be obtained from the QED Lagrangian by the substitution

$$g \rightarrow g', \quad F_{\mu\nu}(x) \rightarrow F'_{\mu\nu}(x) \equiv F_{\mu\nu}(x) - \frac{\hat{L}^\tau}{\pi} \epsilon_{\mu\nu\tau\rho} \int_{-1}^1 \hat{J}^\rho(x + \frac{s}{2}g'^2\hat{L}) ds, \quad (4.33)$$

where \hat{J}^ρ is the $U(1)$ current including the contribution of the scalars.

We see that the S-dual theory actually looks local order by order in g' , and only appears non-local if we sum all orders in g' . In particular, the scale of non-locality in this description is $g'^2\hat{L}$. We can gain a clue as to the origin of the non-locality by rewriting (4.33) using (4.28)

$$F'_{\mu\nu}(x) = F_{\mu\nu}(x) - \frac{\hat{L}^\tau}{\pi} \epsilon_{\mu\nu\tau\rho} \frac{2 \sin \frac{i}{2}g'^2\hat{L} \cdot \partial}{\frac{i}{2}g'^2\hat{L} \cdot \partial} \hat{J}^\rho(x). \quad (4.34)$$

Notice that only even powers of g'^2 enter in the Taylor series expansion of $\frac{\sin \frac{i}{2}g'^2\hat{L} \cdot \partial}{\frac{i}{2}g'^2\hat{L} \cdot \partial}$. It would be interesting to understand this behavior directly by studying string interactions in

the S-dual RR plane wave background (4.5)-(4.7). Note that when the NSNS background (4.2)-(4.4) is transformed into the RR background (4.5)-(4.7) using S-duality, the Regge slope $\tilde{\alpha}'$ of (4.5)-(4.7) is given in terms of the Regge slope α' of (4.2)-(4.4) by $\tilde{\alpha}' = g_s \alpha'$. Using (4.15) and the definition of \hat{L} in (4.31) we see that $\hat{L} = 2\pi\tilde{\alpha}'\mu$ and so is finite in the RR background.

In order to extend the discussion to N D3-brane probes we need to know the S-dual description of the dipole theory that is obtained as a deformation of $\mathcal{N} = 4$ Super Yang-Mills theory with gauge group $U(N)$. Since the gauge fields that correspond to the $U(1)$ center are IR free we can ignore them and consider only the $SU(N)$ dipole theory. It is natural to conjecture that the dual of the lightlike $SU(N)$ dipole theory is given by a prescription similar to (4.33)

$$\begin{aligned}
F_{\mu\nu}(x) &\rightarrow F_{\mu\nu}(x) - \frac{1}{\pi} \epsilon_{\mu\nu\tau\rho} \int_{-1}^1 ds \left\{ \sum_{k=1}^3 \hat{L}_k^\tau \hat{J}_k^\rho(x + \frac{s}{2} g'^2 \hat{L}_k) + \sum_{a=1}^4 \hat{\lambda}_a^\tau \hat{J}_a^\rho(x + \frac{s}{2} g'^2 \hat{\lambda}_a) \right\}, \\
\hat{J}_k^\mu &\equiv \frac{1}{2} (i \bar{Z}_k D^\mu Z_k - i D^\mu \bar{Z}_k Z_k - i Z_k D^\mu \bar{Z}_k + i D^\mu Z_k \bar{Z}_k), \\
\hat{J}_a^\mu &\equiv \frac{1}{2} \sigma^{\mu\alpha}{}_{\dot{\alpha}} \psi_\alpha^a \bar{\psi}_a^{\dot{\alpha}} - \frac{1}{2} \sigma^{\mu\alpha}{}_{\dot{\alpha}} \bar{\psi}_a^{\dot{\alpha}} \psi_\alpha^a
\end{aligned} \tag{4.35}$$

Here we used the notation of section 4.2 and we defined the rescaled dipole vectors of the bosons and fermions similarly to (4.31),

$$\hat{L}_k^\tau \equiv \frac{1}{g'^2} L_k^\tau, \quad \hat{\lambda}_a^\tau \equiv \frac{1}{g'^2} \lambda_a^\tau.$$

The Lie algebra valued \hat{J}_k^μ and \hat{J}_a^μ are the individual contributions of the scalars and fermions to the $su(N)$ current. In (4.35) each contribution to the current enters with a coefficient that is proportional to the dipole vector of the corresponding field. We assume that the dipole vectors \hat{L}_k and $\hat{\lambda}_a$ are all lightlike and pointing in the same direction. We also assume that (4.35) is written in the gauge $A_- = 0$. In this case all the residual gauge transformations are independent of x^- and (4.35) is gauge invariant.

We do not know what should be the modification to the potential of the scalar fields and the Yukawa coupling of the scalars and fermions. It is possible that those interactions

are still given by the $*$ -product modification (4.12) with the unrescaled dipole vectors L_k and λ_a (which are now of order g'^2). Although the scalars and fermions are not expected to transform as dipoles under the “dual” gauge fields (since they are electric but not magnetic dipoles), in the $A_- = 0$ gauge the interactions are gauge invariant even after the modification (4.12).

4.5 Conclusion and Discussion

In this chapter we have argued that particular D-brane probes of plane-wave backgrounds are described by nonlocal field theories. In the case of an NSNS background we have identified the field theory as a lightlike dipole theory and we have verified the statement by an explicit lightcone string computation. In the case of an RR 3-form field strength background we have provided an indirect argument, using S-duality, for the nonlocality of the effective theory on D3-brane probes. This is a more complicated theory and we have conjectured the form of its Lagrangian in (4.33). The nonlocality scale is proportional to g_s and it is obvious from (4.33) that one has to sum up contributions from all orders of string perturbation theory in order to exhibit the nonlocal nature of the interactions. It would be interesting to verify this directly from the solvable plane-wave string theory. Note that, since the nonlocal interactions are in the lightlike direction, we can make the characteristic scale arbitrarily big by a coordinate transformation that rescales x^\pm . The excited string states can therefore decouple safely and, as the field theoretic S-duality suggests, the effective nonlocal field theory can be unitary.

It is interesting to extend these ideas to pp-wave backgrounds with other RR fluxes. For that purpose we adopt the following somewhat heuristic point of view. The dipole theories that we have described in section 4.2 have a correlation between R-symmetry charge and electric flux. In the D3-brane language, every state with $Spin(6)$ transverse angular momentum also behaves as a fundamental string of finite extent. The length of the string is proportional to its angular momentum and the proportionality constants are the dipole vectors \vec{L}_k . In the S-dual nonlocal theories that describe D3-brane probes in pp-waves with

a 3-form RR flux every state with $Spin(6)$ transverse angular momentum also behaves as a D1-brane of finite extent. We can extend this line of thought to other RR-backgrounds. For example, in a background with a 5-form RR field strength $F_{+1234} = F_{+5678}$ (where we use lightcone coordinates $+, -, 1 \dots 8$ as in [81]) and a D5-brane in directions $+, -, 1235$ we should find that open string states attached to the D5-brane that have, say, angular momentum in the 78 plane also behave as a D3-brane that is spread in directions $+, -, 56$ and has a finite volume that is proportional to the angular momentum. This statement is, admittedly, obscure and it would be interesting to elucidate such a theory further.

Another possible application of the ideas presented in this chapter is to M(atrrix)-theory [115]. As we have mentioned earlier, the M(atrrix)-theory Hamiltonian for M-theory is 0+1D supersymmetric Yang-Mills quantum mechanics [116]-[119]. The standard derivation of weakly coupled type-IIA string theory from M(atrrix)-theory [120]-[122] requires understanding of the strong coupling limit of 1+1D $\mathcal{N} = 8$ Super Yang-Mills theory. Dipole theories naturally appear as M(atrrix) models of Melvin spaces [104] (see also [102, 103]). The relevant M(atrrix) models are dipole theories that are deformations of 1+1D $\mathcal{N} = 8$ Super Yang-Mills theory. Therefore understanding the strong coupling limit of dipole theories could prove beneficial for deriving a weakly coupled string theory descriptions of Melvin spaces. (A string theory description for Melvin backgrounds has been given in [123] but it has a dilaton that is not bounded.) Perhaps nonlocal worldsheet theories will play a role in such a description. (See [124, 125] for other ideas regarding nonlocal worldsheet theories.)

We would also like to mention another new kind of nonlocal theory that appears on D3-brane probes in certain backgrounds with strong NSNS flux [126]-[128]. It is a very intriguing nonlocal field theory that is not translationally invariant and is described as a gauge theory on a noncommutative space with a varying noncommutativity parameter.

Thus, as we have seen in this chapter as well as the previous chapter, non-local theories associated with noncommutative deformations seem to be fairly generic in string theory. In this chapter, by considering light-like limits of dipole theories, we have begun to consider novel modifications of space-time geometry associated with non-trivial physics in Lorentzian signature. The next chapter will begin to approach physics questions that are completely

unique to Lorentzian manifolds, that of closed time-like curves and the radical notion of holography.

Chapter 5

Holography and the Gödel Universe in String Theory

In the final chapter of this dissertation, we begin to consider the question of how string theory and quantum gravity might address issues that are unique to Lorentzian signature space-times. In particular, we analyze the structure of supersymmetric Gödel-like cosmological solutions of string theory. Just as the original four-dimensional Gödel universe, these solutions represent rotating, topologically trivial cosmologies with a homogeneous metric and closed timelike curves. As we have learned in the case of *AdS* space [80], the notion of holography [129] seems to be key in understanding the physics of non-trivial Lorentzian space-times. The work presented in this chapter was done in collaboration with Ed Boyda, Surya Ganguli, and Petr Hořava [159].

We focus on the “phenomenological” aspects of holography, and identify the preferred holographic screens associated with inertial comoving observers in Gödel universes. We find that holography can serve as a chronology protection agency: The closed timelike curves are either hidden behind the holographic screen, or broken by it into causal pieces. In fact, holography in Gödel universes has many features in common with de Sitter space, suggesting that Gödel universes could represent a supersymmetric laboratory for addressing the

conceptual puzzles of de Sitter holography. Then we initiate the investigation of “microscopic” aspects of holography of Gödel universes in string theory. We show that Gödel universes are T-dual to pp-waves, and use this fact to generate new Gödel-like solutions of string and M-theory by T-dualizing known supersymmetric pp-wave solutions.

5.1 Introduction and Summary

Many long-standing conceptual questions of quantum gravity, and even of classical general relativity, are finding their answers in string theory. Among the most notable examples are various classes of supersymmetric timelike singularities, or the microscopic explanation of Bekenstein-Hawking entropy for a class of configurations controllable by spacetime supersymmetry. On the other hand, many puzzles of quantum gravity still remain unanswered. In particular, the role of time in cosmological, and other time-dependent, solutions of string theory still defies any systematic understanding.

While many crucial questions of quantum gravity are associated with high spacetime curvature or with cosmological horizons, some puzzles become apparent already in spacetimes with very mild curvature, no horizons, and even trivial topology. How can the low-energy classical relativity fail to represent a good approximation to quantum gravity for small curvature and in the absence of horizons? Arguments leading to the holographic principle [129] indicate that general relativity misrepresents the true degrees of freedom of quantum gravity, by obscuring the fact that they are secretly holographic. In those instances where string theory has been successful in resolving puzzles of quantum gravity, it has done so by identifying the correct microscopic degrees of freedom, which frequently are poorly reflected by the naive (super)gravity approximation. In this paper we investigate an example in which holography suggests a very specific dramatic modification of the degrees of freedom in quantum gravity already at very mild curvatures, in a homogeneous and highly supersymmetric cosmological background.

Historically, microscopic holography in string theory has been relatively easier to understand for solutions with a “canonical” preferred holographic screen which is observer-

independent, and typically located at asymptotic infinity. Holography in AdS spaces is a prime example of this. On the other hand, cosmological backgrounds in string theory require an understanding of holography in more complicated environments, which may not exhibit canonical, observer-independent preferred screens at conformal infinity. Here, the prime example is given by de Sitter space: When viewed from the perspective of an inertial observer living in the static patch, the preferred holographic screen in de Sitter space is most naturally placed at the cosmological horizon. This leads to the fascinating idea of observer-dependent holographic screens, associated with a finite number of degrees of freedom accessible to the observer (for more details, see e.g. [130, 131, 132, 133, 134]; see also [135, 136] for a complementary point of view on de Sitter holography that uses other preferred screens, not associated with an inertial observer).

Of course, string theory promises to be a unified theory of gravity and quantum mechanics, but it is at present unclear how it manages to reconcile the general relativistic concept of time (notoriously difficult because of spacetime diffeomorphism invariance) with the quantum mechanical role of time as an evolutionary Hamiltonian parameter. Again, this problem becomes somewhat trivialized in the presence of supersymmetry, but persists in all but the most trivial time-dependent backgrounds of string theory.

In this chapter, we analyze a class of supersymmetric solutions of string theory and M-theory, which – at least in the classical supergravity approximation – are described by geometries with no global time function. In particular, we focus our attention on string theory analogs of Gödel’s universe. Gödel’s original solution [137] is a homogeneous rotating cosmological solution of Einstein’s equations with pressureless matter and negative cosmological constant, which played an important role in the conceptual development of general relativity. Recently, a supersymmetric generalization of Gödel’s universe has been discussed in a remarkable paper by Gauntlett *et al.* [138], who classified all supersymmetric solutions of five-dimensional supergravity with eight supercharges, and found a maximally supersymmetric Gödel-like solution that can be lifted to a solution of M-theory with twenty Killing spinors. The existence of this solution was also noticed previously by Tseytlin, see Footnote 26 of [139]. It is worth stressing that the Gödel universe of M-theory is

time-orientable: There is an invariant notion of future and past lightcones, at each point in spacetime. Also, there is a global time *coordinate* t , and in fact $\partial/\partial t$ is an everywhere time-like Killing vector (in effect, making supersymmetry possible). However, t is not a global time *function*: The surfaces of constant t are not everywhere spacelike.¹ Actually, the solution cannot be foliated by everywhere-spacelike surfaces at all – the classical Cauchy problem is always ill-defined in this spacetime. It is hard to imagine how such an apparently pathological behavior of global time could be compatible with the conventional role of time in the Hamiltonian picture of quantum mechanics. Indeed, this solution turns out to have classical pathologies: Just as Gödel’s original solution, the supersymmetric Gödel metric allows closed timelike curves, seemingly suggesting either the possibility of time travel (cf. [141]) or at least grave causality problems.

These classical pathologies could imply that the Gödel solution, despite its high degree of supersymmetry, stays inconsistent even in full string or M-theory. There are of course pathological solutions of Einstein’s equations whose problems do not get resolved in string theory, with the negative-mass Schwarzschild black hole being one example.

However, there are reasons why one might feel reluctant to discard this solution as manifestly unphysical, despite the sicknesses of the classical metric: This solution is homogeneous, its curvature can be kept small everywhere (in particular, there are no singularities and no horizons), and the solution is highly supersymmetric. It is also impossible to eliminate the closed timelike curves by going to a universal cover – indeed, the Gödel solution is already topologically trivial.²

We feel that any solution should be presumed consistent until proven otherwise, and this will be our attitude towards the Gödel solution in this chapter. Our aim will be to analyze holographic properties of the supersymmetric Gödel solution in string theory. The solution is remarkably simple, and as we will see in Section 5, turns out to be related by duality to the

¹See, e.g., [140] for a detailed discussion of the distinction between a global time coordinate and a global time function.

²This should be contrasted with the case of solutions with “trivial” (in the sense of Carter [142]) closed timelike curves, such as those in the flat Minkowski spacetime with time compactified on \mathbf{S}^1 , where the closed timelike curves can be eliminated by lifting the solution to its universal cover.

solvable supersymmetric pp-wave backgrounds much studied recently. However, before we attempt the analysis of “microscopic” holography in string theory, we will first adopt a more “phenomenological” approach as pioneered by Bousso [143] (see [131, 144] for reviews), and analyze the structure of preferred holographic screens implied by the covariant prescription [143] for their identification in classical (super)gravity solutions. This “phenomenological” analysis leads to valuable hints, indicating how the problem of closed timelike curves may be resolved in the Gödel universe. Indeed, we will claim that the apparent pathologies of the semiclassical supergravity solution can be resolved when holography is properly taken into account. Semiclassical general relativity without holography is not a good approximation of this solution, despite its small curvature, absence of horizons, and trivial spacetime topology.

Notice also that homogeneity of the Gödel solution makes things at least superficially worse: It implies that there are closed timelike curves through every point in spacetime. However, these closed timelike curves are also in a sense (to be explained below) topologically “large.” Our analysis of the structure of holographic screens in this geometry reveals an intricate system of observer-dependent preferred holographic screens, which always carve out a causal part of spacetime, and effectively screen all the closed timelike curves and hide any violations of causality from the inertial observer. In fact, the causal structure of the part of spacetime carved out by the screen is precisely that of an AdS space, cut off at some finite radial distance.

The preferred holographic screens in the Gödel universe are very much like the screens associated with the inertial observers in the static patch of de Sitter space. First of all, they are associated with the selection of an observer (and therefore represent “movable,” non-canonical screens, not located at conformal infinity). Moreover, they are compact, implying a finite covariant bound on entropy and – in the strong version of the holographic principle – a finite number of degrees of freedom associated with any inertial observer. Thus, the Gödel universe should serve as a useful supersymmetric laboratory for addressing some of the conceptual puzzling issues of de Sitter holography.

The results of our “phenomenological” analysis of holography also reveal the importance, for cosmological spacetimes, of a local description of physics as associated with an observer

inside the universe. It is not sensible to pretend that the observer stays at asymptotic infinity, and observes only elements of the traditionally defined S-matrix (or some suitable analogs thereof). Clearly, this only stresses the need for a conceptual framework defining more environmentally-friendly, “cosmological” observables as associated with cosmological observers in string theory.

This chapter is organized as follows. In Section 5.2, we set the stage by reviewing and analyzing Gödel’s cosmological solution $\mathcal{G}_3 \times \mathbf{R}$ of Einstein’s gravity in four space-time dimensions. Despite its simplicity, this solution already exhibits all the crucial issues of our argument. We apply Bousso’s prescription for the covariant holographic screens, and find screens that are observer-dependent, compact, and causality-preserving. In addition, we establish connection with holography in AdS spaces: Gödel’s solution can be viewed as a member of a two-parameter moduli space of homogeneous solutions of Einstein’s equations with trivial spacetime topology, with $AdS_3 \times \mathbf{R}$ also in this moduli space. We show that under the corresponding deformation the observer-dependent preferred holographic screens of Gödel’s universe recede to infinity and become the canonical holographic boundary of $AdS_3 \times \mathbf{R}$. In Section 5.3 we move on to the supersymmetric Gödel universe of M-theory, which can be written as $\mathcal{G}_5 \times \mathbf{R}^6$. First we analyze the \mathcal{G}_5 part of the geometry as a solution of minimal $d = 5$ supergravity, study in detail the structure of geodesics in this solution and use it to determine the preferred holographic screens, and show how chronology can be protected by holography. Then we extend our analysis to the full $\mathcal{G}_5 \times \mathbf{R}^6$ Gödel geometry in M-theory. Section 5.4 points out remarkable analogies between holography in the supersymmetric Gödel universe and holography in de Sitter space. In Section 5.5, we embark on the analysis of “microscopic” duality of Gödel universes in string theory. First, we compactify the M-theory solution on \mathbf{S}^1 to obtain a Gödel solution of Type IIA superstring theory, and show that upon further \mathbf{S}^1 compactification the Type IIA Gödel universe is T-dual to a supersymmetric Type IIB pp-wave, which can be obtained as the Penrose limit of the intersecting D3-D3 system. We point out that this Gödel/pp-wave T-duality is a much more general phenomenon, and can be used to construct new Gödel universes in string and M-theory by T-dualizing known pp-waves. The relation to pp-waves

is just one aspect of the remarkable degree of solvability of Gödel solutions in string theory. We intend to present a more detailed analysis of “microscopic” aspects of holography in the Gödel universes of string and M-theory elsewhere [145]. Finally, in Section 5.6 we summarize some geometric properties of the supersymmetric Gödel solutions.

5.2 Holography in Gödel's Four-Dimensional Universe

5.2.1 Gödel's solution

In 1949, on the occasion of Albert Einstein's 70th birthday, Kurt Gödel presented a rotating cosmological solution [137] of Einstein's equations with negative cosmological constant and pressureless matter; this solution is topologically trivial and homogeneous but exhibits closed timelike curves. Our exposition of Gödel's solution follows [137, 1].

The spacetime manifold of this solution has the trivial topology of \mathbf{R}^4 , which we will cover by a global coordinate system (τ, x, y, z) . The metric factorizes into a direct sum of the (trivial) metric dz^2 on \mathbf{R} and a nontrivial metric on \mathbf{R}^3 ,

$$ds_4^2 = ds_3^2 + dz^2, \quad (5.1)$$

where

$$ds_3^2 = -d\tau^2 + dx^2 - \frac{1}{2}e^{4\Omega x}dy^2 - 2e^{2\Omega x}d\tau dy. \quad (5.2)$$

This class of solutions is characterized by a rotation parameter Ω . We will refer to the manifold \mathbf{R}^3 equipped with the non-trivial part 5.2 of Gödel's metric as \mathcal{G}_3 . Thus, in our notation, Gödel's universe is $\mathcal{G}_3 \times \mathbf{R}$. The metric on \mathcal{G}_3 has a four-dimensional group of isometries. The geometry exhibits dragging of inertial frames, associated with rotation. The full four-dimensional geometry solves Einstein's equations, with the value of the cosmological constant and the density of pressureless matter both determined by the rotation parameter Ω ,

$$\rho = \frac{\Omega^2}{2\pi G_N}, \quad \Lambda = -2\Omega^2. \quad (5.3)$$

Historically, this solution was instrumental in the discussion of whether or not classical general relativity satisfies Mach's principle (see, e.g., [1], Sect. 12.4).

While the homogeneity of Gödel's universe is (almost) manifest in the coordinate system used in 5.2, the rotational symmetry of ds_3^2 around any point in space becomes more obvious in cylindrical coordinates (t, r, ϕ) , in which the metric takes the following form,

$$ds_3^2 = -dt^2 + dr^2 - \frac{1}{\Omega^2} (\sinh^4(\Omega r) - \sinh^2(\Omega r)) d\phi^2 - \frac{2\sqrt{2}}{\Omega} \sinh^2(\Omega r) dt d\phi. \quad (5.4)$$

Indeed, $\partial/\partial\phi$ is a Killing vector, of norm squared

$$\left| \frac{\partial}{\partial\phi} \right|^2 = \frac{1}{\Omega^2} (1 - \sinh^2(\Omega r)) \sinh^2(\Omega r). \quad (5.5)$$

The orbits of this Killing vector are closed, and become closed timelike curves for $r > r_0$,

$$r_0 = \frac{1}{\Omega} \operatorname{arcsinh}(1) \equiv \frac{1}{\Omega} \ln(1 + \sqrt{2}). \quad (5.6)$$

We will call the surface of $r = r_0$ the *velocity-of-light surface*; the null geodesics emitted from the origin in this coordinate system reach the velocity-of-light surface in a finite affine parameter, and then spiral back to the origin where they refocus, again in finite affine parameter.

The homogeneity of the solution implies that there are closed timelike curves through every point in spacetime. Note that in a well-defined sense all the closed timelike curves are topologically “large”: In order to complete a closed timelike trajectory starting at any point P , one has to travel outside of the velocity-of-light surface (as defined by an observer at P) before being able to return to P along a causal curve. This fact will play an important role in our argument for the holographic resolution of the problem of closed timelike curves below. Notice also that none of the closed timelike curves is a geodesic, and that the closed timelike curves cannot be trivially eliminated by a lift to the universal cover: The manifold is already topologically trivial.

Gödel's universe represents a solution with a good timelike Killing vector (indeed, $\partial/\partial t$ is Killing and everywhere timelike), which however cannot be used to define a universal time function: The slices of the foliation by surfaces of constant t are not everywhere spacelike. The classical Cauchy problem is always globally ill-defined for this geometry.

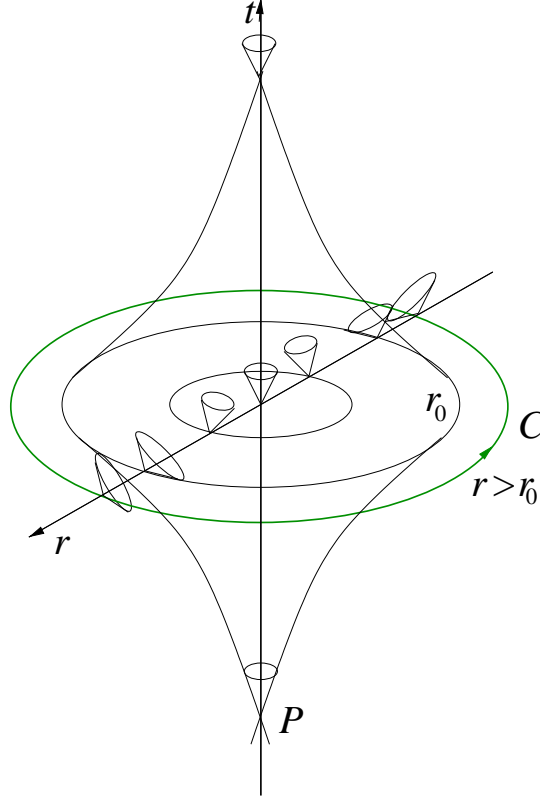


Figure 5.1: The geometry of the three-dimensional part \mathcal{G}_3 of Gödel's universe, with the flat fourth dimension z suppressed. Null geodesics emitted from the origin P follow a spiral trajectory, reach the velocity-of-light surface at the critical distance r_0 , and spiral back to the origin in finite affine parameter. The curve C of constant $r > r_0$ tangent to $\partial/\partial\phi$ is an example of a closed timelike curve. A more detailed version of this picture appears in Hawking and Ellis [1].

5.2.2 Preferred holographic screens in Gödel's universe

We now apply Bousso's phenomenological framework for holography [143, 131, 144] to Gödel's universe. We identify its preferred holographic screens, associated with particular observers as follows:

Consider a geodesic observer comoving with the distribution of dust in Gödel's universe (and placed at the origin $r = 0$ of our coordinate system without loss of generality). Imagine

that the observer sends out lightrays in all directions from the origin at some fixed time, say $t = 0$. These lightrays will at first expand – i.e., the surfaces that they reach in some fixed affine parameter λ will grow in area, at least for small enough values of λ . The preferred holographic screen will be reached when we reach the surface of maximal area (or maximal geodesic expansion).

Alternatively, one can follow *incoming* lightrays into their past, until reaching the surface where the geodesics no longer expand. This is again the location of the preferred screen \mathcal{B} . The preferred screen \mathcal{B} can then be used to impose a covariant bound on the entropy inside the region of space surrounded by \mathcal{B} [143], which should not exceed one-fourth of the area of \mathcal{B} in Planck units.

We will first analyze the three-dimensional part \mathcal{G}_3 of Gödel's solution, which contains much of the nontrivial geometry. Even though all the geodesics of Gödel's universe are known [147], one can in fact use the symmetries of \mathcal{G}_3 to determine the location of the screen without any explicit knowledge of the geodesic curves. Since \mathcal{G}_3 is rotationally invariant in ϕ , all the null geodesics emitted from the origin will reach the same radial distance $r(\lambda)$ within the same affine parameter (assuming that we use a rotationally invariant normalization of λ for geodesics emitted in different directions from the origin), and also for the same global time coordinate t . Thus, to determine the surface of maximal geodesic expansion, we can just evaluate the area of the surfaces of constant r and t (in our case of course one-dimensional),

$$A = \frac{2\pi}{\Omega} \sinh(\Omega r) \sqrt{1 - \sinh^2(\Omega r)}, \quad (5.7)$$

and maximize it as a function of r . This very simple calculation yields a preferred screen \mathcal{H} that is isomorphic to a cylinder of constant $r = r_{\mathcal{H}}$ and any t , with

$$r_{\mathcal{H}} = \frac{1}{\Omega} \operatorname{arcsinh} \left(\frac{1}{\sqrt{2}} \right). \quad (5.8)$$

Of course, this screen is observer-dependent, in this case associated with the comoving inertial observer located at the origin for all values of t . Other comoving inertial observers would see different but isomorphic screens, in a pattern similar to the structure of cosmological horizons associated with inertial observers in de Sitter space.

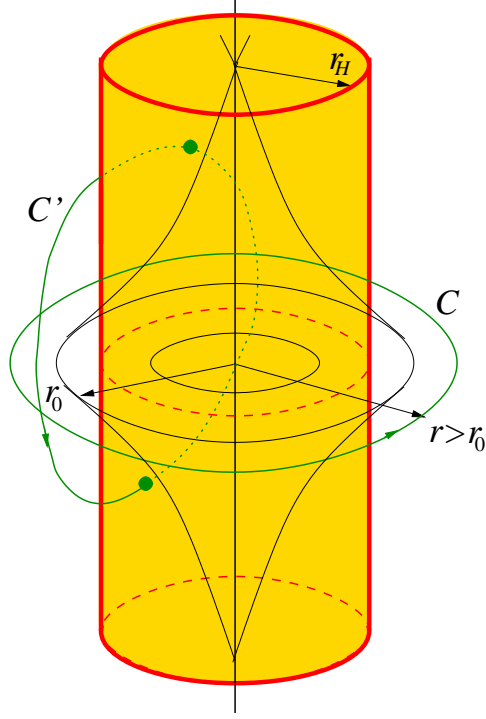


Figure 5.2: The geometry of our preferred holographic screen in Gödel's universe, as defined by the inertial observer following the comoving geodesic at the origin of spatial coordinates. The translationally invariant dimension z is again suppressed. Two closed timelike curves are indicated: One, C , at constant value of $t = 0$ and $r > r_0$ is outside of the preferred screen, while another, C' , passes through the origin at $t = 0$ and intersects the screen in two, causally connected points.

One can take advantage of the rotational symmetry of the solution, and visualize the location of the preferred screen using a spacetime diagram of the type introduced by Bousso [143] (see Figure 3). This diagram suppresses the dimension of rotational symmetry ϕ , and its points represent (in our case one-dimensional) orbits of the rotation group, i.e., surfaces of constant r and t . For each such surface, one can define the total of four lightsheets: Two oriented forward in time, and two oriented backward. In generic points of the diagram, two of these lightsheets will be non-expanding. At each point of the Bousso diagram one can draw a wedge pointing in the direction of non-expanding lightsheets. These wedges then

point in the direction of the preferred holographic screen.

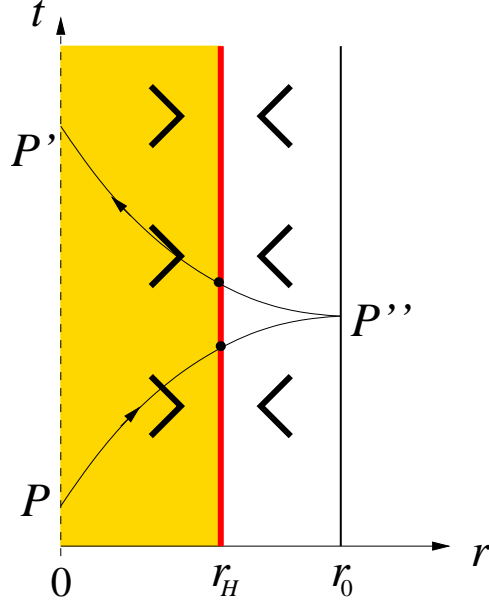


Figure 5.3: The Bousso diagram for the \mathcal{G}_3 part of the Gödel universe metric, with the angular coordinate ϕ suppressed, and the structure of non-expanding lightsheets indicated by the bold wedges. The preferred holographic screen is at the finite value $r_{\mathcal{H}}$ of the radial coordinate r , strictly smaller than the location of the velocity-of-light surface at r_0 . A null geodesic sent from P would reach the velocity-of-light surface at P'' in a finite affine parameter, and refocus again at the spatial origin in P' .

One can directly verify that our preferred holographic screen satisfies the defining property

$$\theta = 0, \quad (5.9)$$

where θ is the expansion parameter defined for a spacelike codimension-two surface B (in any spacetime with coordinates x^μ) as

$$\theta = h^{\mu\nu} D_\mu \zeta_\nu, \quad (5.10)$$

with ζ_μ the light-like covector orthogonal to B (smoothly but arbitrarily extended to some neighborhood of B), D_μ is the covariant derivative, and $h_{\mu\nu}$ is the induced metric on B .

The most convenient way of identifying the surface of $\theta = 0$ in Gödel's universe is to use as ζ the vector tangent to the congruence of null geodesics emitted by the observer at the origin. An explicit calculation confirms in this case that θ is proportional to $\partial_r g_{\phi\phi}$, and therefore vanishes at the surface of $r = r_{\mathcal{H}}$.

The metric induced on the preferred holographic screen \mathcal{H} is of signature $(-+)$, everywhere nonsingular:

$$ds_{\mathcal{H}}^2 = -dt^2 + \frac{1}{4\Omega^2}d\phi^2 + \frac{\sqrt{2}}{\Omega}d\phi dt \quad (5.11)$$

with $0 \leq \phi \leq 2\pi$. The preferred holographic screen carves out a cylindrical compact region of spacetime (which we will call the *holographic region*) in the \mathcal{G}_3 part of Gödel's universe, centered on the comoving inertial observer at the origin. This region contains no closed timelike curves, as can be easily demonstrated by noticing that the causal structure of the holographic region is identical to that of a cylindrical portion of (the universal cover of) AdS_3 . The closed timelike curves of the full \mathcal{G}_3 geometry fall into two categories: Either they stay completely outside of the holographic region, or they enter it and leave it again after traveling a causal trajectory within the holographic region.

Preferred screens in $\mathcal{G}_3 \times \mathbf{R}$

The full Gödel universe is of the direct product form $\mathcal{G}_3 \times \mathbf{R}$. The presence of the extra, translationally-invariant dimension parametrized by z actually implies a richer structure of preferred screens than the one we just found in the \mathcal{G}_3 factor. This is in fact a preview of what we will find in the next section in the case of supersymmetric Gödel solutions in M-theory and string theory: Those solutions typically also contain extra flat dimensions.

First of all, there is one preferred screen that can be easily identified: The three-dimensional surface $\mathcal{H} \times \mathbf{R}$, where \mathcal{H} is the preferred screen associated with the observer at the spatial origin in \mathcal{G}_3 , and \mathbf{R} is the extra coordinate z , clearly satisfies the zero-expansion condition 5.9. Thus, by definition, this surface $\mathcal{H} \times \mathbf{R}$ is a preferred screen. This screen is observer-dependent, and the observer associated with it can be thought of either as a string wrapped around z or as a more traditional observer “delocalized” along z , each localized

at the origin of coordinates in \mathcal{G}_3 . Unless we compactify z on \mathbf{S}^1 , the overall area of this translationally-invariant screen is of course infinite, but the screen still has a finite “area density” per unit distance along z .

Alternatively, one can ask what is the preferred screen associated with an localized inertial observer in $\mathcal{G}_3 \times \mathbf{R}$. If one follows null geodesics emitted from (or converging onto) a point in $\mathcal{G}_3 \times \mathbf{R}$ where the observer is located, one finds that the surface of maximal geodesic expansion is at a finite distance from the observer in all space directions including z . This compact, translationally-noninvariant screen is completely contained within the velocity-of-light surface as defined by the observer.

For either of these two classes of screens in $\mathcal{G}_3 \times \mathbf{R}$, all closed timelike curves are again either hidden outside of the screen or broken by it into causal observable pieces.

Covariant entropy bounds and screen complementarity

The existence of preferred screens, and the structure of the Bousso diagram for Gödel's universe imply a holographic entropy bound on the amount of entropy through any spatial slice of the compact holographic region associated with each screen. This entropy is limited by one fourth of the area of the screen measured in Planck units. Our screen is neither at conformal infinity, nor located at a horizon. The closest analog would be the preferred holographic screen located at the equator of the Einstein static universe. Just as in that case, the holographic screen of Gödel's universe can be used to bound the entropy in either direction normal to the screen. In particular, the lightrays that start at the screen and travel in the direction of larger values of r refocus at the velocity-of-light surface, and then travel back again to the screen. This is rather reminiscent of the behavior of lightrays in Einstein's static universe: lightrays emitted from one pole of the spatial sphere reach the screen at the equator and travel to the other hemisphere, refocus at the opposite pole, and travel back to the screen and then to the point they were originally emitted from.

The strong version of the holographic principle suggests that the compact holographic screen implies a finite bound on the number of degrees of freedom effectively accessible to

the inertial observer. The good causal structure of the holographic region associated with that observer may suggest that the quantum mechanics of this finite number of degrees of freedom could be well-defined, and screened from the acausal behavior outside of the velocity-of-light surface by a screen complementarity principle.

Of course, one may find the very definition of entropy in spacetimes with closed timelike curves somewhat problematic. However, in the case of Gödel's universe all that matters for our argument is the region strictly below the velocity-of-light surface. One can in principle imagine cutting Gödel's solution off at some finite r larger than $r_{\mathcal{H}}$ but smaller than r_0 , and replacing the outside with some causal geometry. The covariant entropy bound can then be safely applied to the holographic region, without any possible conceptual difficulties with the definition of entropy in the presence of closed timelike curves.

The intricate structure of compact preferred screens associated with the observers in Gödel's universe suggests that holography may be the correct, causal way of thinking about this geometry without modifying it. However, one is forced to replace the naive “metaobserver” perspective of the geometry by a system of local observers, each of which sees a causal region screened from the rest of the naive classical geometry by the preferred holographic screen. Each individual observer would only have access to a finite amount of degrees of freedom associated with the corresponding holographic region. Within this finite number of degrees of freedom, causality and quantum mechanics would be protected.

In this chapter we will not discuss non-inertial observers attempting to travel along closed timelike curves. In the spirit of Hawking's original chronology protection conjecture [148], one may expect a large backreaction from the geometry that can protect the solution from such observers.

5.2.3 Gödel's universe as deformed AdS_3 and holography

It is useful to embed our discussion of Gödel's universe into a broader framework. Consider all spacetime-homogeneous metrics of the Gödel type. It has been shown [149] that this family of metrics is parametrized by two parameters, Ω and m , with the metric

given by

$$ds^2 = - \left(dt + \frac{4\sqrt{2}\Omega}{m^2} \sinh^2 \left(\frac{mr}{2} \right) d\phi \right)^2 + \frac{1}{m^2} \sinh^2(mr) d\phi^2 + dr^2 + dz^2, \quad (5.12)$$

with $\Omega \in \mathbf{R}$ and $m^2 \in \mathbf{R}$. For $m^2 = 4\Omega^2$, we recover Gödel's metric 5.2 . On the other hand, for $m^2 = 8\Omega^2$ we get the direct-product metric on $AdS_3 \times \mathbf{R}$ [150] . Notice also that the metric simplifies in the limit of $m \rightarrow 0$ keeping Ω fixed; this metric has been analyzed by Som and Raychaudhuri [151] , and is in fact a closer analog of the string theory Gödel universe than Gödel's solution itself.

Since all the solutions in 5.12 are rotationally invariant, we can easily identify the preferred screens for this entire family of metrics using the same symmetry argument as in Gödel's universe itself. The holographic screens \mathcal{H} of the non-trivial three-dimensional part of 5.12 are now located at

$$r_{\mathcal{H}} = \frac{2}{m} \operatorname{arcsinh} \left(\left(\frac{16\Omega^2}{m^2} - 2 \right)^{-1/2} \right). \quad (5.13)$$

Thus, for $m^2 < 8\Omega^2$, the screen is at a finite value of $r_{\mathcal{H}}$, and as we approach the $AdS_3 \times \mathbf{R}$ limit it recedes to infinity and becomes the canonical holographic screen of AdS_3 . This connection with AdS_3 leads to a particularly intriguing way of thinking about holography of this family of solutions in terms of breaking conformal invariance on the holographic screen of AdS_3 once we move away from the AdS_3 limit.

Clearly, our observation that preferred holographic screens can either screen closed time-like curves or break them up into causal pieces is not restricted to homogeneous spacetimes. An example of the same phenomenon in an inhomogeneous solution can be easily found: Consider the classic cylindrically symmetric inhomogeneous solution with closed timelike curves found in 1937 by van Stockum, [152] , which in the cylindrical coordinates takes the form

$$ds^2 = -dt^2 - 2\Omega r^2 d\phi dt + r^2(1 - \Omega^2 r^2)d\phi^2 + e^{-\Omega^2 r^2}(dz^2 + dr^2). \quad (5.14)$$

It is straightforward to show that the preferred holographic screen as defined by the inertial

observer located at the origin is again compact and shields the closed timelike curves from the observer, just as in the case of the homogeneous Gödel universe.

5.3 Holography in Supersymmetric Gödel Universes

The Gödel solution of M-theory found in [138] has a direct product form $\mathcal{G}_5 \times \mathbf{R}^6$, where the non-trivial five-dimensional part \mathcal{G}_5 represents a maximally supersymmetric solution of minimal supergravity in five dimensions. The underlying spacetime of \mathcal{G}_5 is topologically trivial, isomorphic to \mathbf{R}^5 . Again, just as in the case of Gödel's four-dimensional solution, much of the nontrivial structure of the solution is carried in this five-dimensional factor \mathcal{G}_5 , which plays a role analogous to \mathcal{G}_3 of the previous section. We will therefore study holography of this five-dimensional solution first.

5.3.1 Holography in the Gödel universe of $N = 1$ $d = 5$ supergravity

The five-dimensional Gödel geometry \mathcal{G}_5 is a maximally supersymmetric, topologically trivial, homogeneous solution of minimal five-dimensional supergravity [138]. We introduce generic coordinates X^μ , $\mu = 0, \dots, 4$ on \mathbf{R}^5 , but we will soon specialize to several specific coordinate systems. The minimal $d = 5$ supergravity contains an Abelian gauge field A_μ whose field strength $F_{\mu\nu}$ we normalize such that the Lagrangian has the following form,

$$\mathcal{L}_5 = \frac{1}{2\kappa_5^2} \int d^5 X \left(R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \dots \right), \quad (5.15)$$

where the “...” stand for a Chern-Simons self-interaction of the gauge field and for fermionic terms.

The Gödel solution takes the form of a fibration over the flat Euclidean \mathbf{R}^4 with fibers isomorphic to \mathbf{R} and with a simple twist, which in a Cartesian coordinate system t, x_i , $i = 1, \dots, 4$, can be written as

$$ds^2 = -(dt + \beta\omega)^2 + \sum_{i=1}^4 dx_i^2, \quad (5.16)$$

$$F = 2\sqrt{3}\beta J, \quad (5.17)$$

with the twist one-form ω given by

$$\omega = x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3 \equiv J_{ij} x_i dx_j, \quad (5.18)$$

and $J_{12} = -J_{21} = J_{34} = -J_{43} = 1$ a preferred Kähler form on \mathbf{R}^4 . In 5.16, β is a constant rotation parameter, of mass dimension one. Without any substantial loss of generality, we will assume β to be positive.

As remarked in [138], this solution is homogeneous, and in fact has a nine-dimensional group of bosonic isometries. The Killing vectors are given by

$$P_0 = \partial_t, \quad (5.19)$$

$$P_i = \partial_i - \beta J_{ij} x_j \partial_t, \quad (5.20)$$

$$L = x_1 \partial_2 - x_2 \partial_1 + x_3 \partial_4 - x_4 \partial_3, \quad (5.21)$$

$$R_1 = x_1 \partial_2 - x_2 \partial_1 - x_3 \partial_4 + x_4 \partial_3, \quad (5.22)$$

$$R_2 = x_1 \partial_3 - x_3 \partial_1 + x_2 \partial_4 - x_4 \partial_2, \quad (5.23)$$

$$R_3 = x_1 \partial_4 - x_4 \partial_1 + x_3 \partial_2 - x_2 \partial_3, \quad (5.24)$$

where $\partial_i = \partial/\partial x_i$. The commutation relations of this bosonic symmetry algebra are

$$[R_\alpha, R_\beta] = 2\epsilon_{\alpha\beta\gamma} R_\gamma, \quad [L, R_\alpha] = 0, \quad [P_i, P_j] = 2\beta J_{ij} P_0. \quad (5.25)$$

Here $\alpha, \beta, \dots = 1, 2, 3$ go over a basis of anti-selfdual two-tensors in \mathbf{R}^4 . R_α and L act on the momenta P_i as rotations. Thus, we find that the symmetry algebra of the Gödel universe \mathcal{G}_5 is given by the semidirect product $H(2) \ltimes (SU(2) \times U(1))$, where $H(2)$ is the Heisenberg algebra with five generators.³

While the translation symmetries P_i of the solution are almost manifest in the cartesian coordinates t, x_i , the rotation symmetries are rather obscure. It is therefore convenient to

³As we will see in Section 4, the remarkable similarity between this symmetry algebra and a pp-wave symmetry algebra is not a coincidence: When lifted to string theory, the Gödel solution is actually T-dual to a supersymmetric pp-wave! Notice, however, that in the symmetry algebra of \mathcal{G}_5 , the central extension generator P_0 of the Heisenberg algebra is represented by a timelike Killing vector, while in the pp-wave it would be null. One can actually show by a direct calculation that the five-dimensional Gödel universe (or the string theory lifts thereof to be studied below) does not admit any covariantly constant null vectors, which proves that it is not “secretly” a pp-wave in unusual coordinates.

introduce a new coordinate system. First, we introduce a pair of polar coordinates, one in each of the two main planes of rotation,

$$x_1 = r_1 \cos \phi_1, \quad x_3 = r_2 \cos \phi_2, \quad (5.26)$$

$$x_2 = r_1 \sin \phi_1, \quad x_4 = r_2 \sin \phi_2. \quad (5.27)$$

In these “bipolar” coordinates, the metric becomes

$$\begin{aligned} ds^2 = & -dt^2 - 2\beta(r_1^2 d\phi_1 + r_2^2 d\phi_2)dt + dr_1^2 + dr_2^2 - 2\beta^2 r_1^2 r_2^2 d\phi_1 d\phi_2 \\ & + r_1^2(1 - \beta^2 r_1^2)d\phi_1^2 + r_2^2(1 - \beta^2 r_2^2)d\phi_2^2. \end{aligned} \quad (5.28)$$

The non-Abelian part of the rotation symmetries becomes manifest in spherical coordinates $(r, \phi_1, \phi_2, \vartheta)$, with $\vartheta \in [0, \pi/2)$,

$$x_1 + ix_2 = r e^{i\phi_1} \cos \vartheta, \quad (5.29)$$

$$x_3 + ix_4 = r e^{i\phi_2} \sin \vartheta, \quad (5.30)$$

which bring the metric to the following form,

$$ds^2 = -\left(dt + \frac{\beta r^2}{2}\sigma_3\right)^2 + dr^2 + r^2 d\Omega_3^2. \quad (5.31)$$

Here $d\Omega_3^2$ is the standard unit-volume metric on \mathbf{S}^3 , and σ_3 is one of the right-invariant one-forms on $SU(2)$,

$$\sigma_3 = 2(\cos^2 \vartheta d\phi_1 + \sin^2 \vartheta d\phi_2). \quad (5.32)$$

It is clear from this expression for the metric that even though the solution does not exhibit the full $SO(4) \sim SU(2) \times SU(2)$ rotation symmetry in \mathbf{R}^4 , the non-zero rotation parameter β keeps the right $SU(2)$ (together with a $U(1)$ subgroup of the left $SU(2)$) unbroken.

It was also noted in [138] that the Gödel universe \mathcal{G}_5 preserves all eight supersymmetries of minimal $d = 5$ supergravity. Thus, the bosonic symmetry algebra 5.25 will extend to a superalgebra with eight supercharges Q . It is natural to split Q into two four-component

spinors, Q^\pm . In this notation, the (anti)commutation relations of the full symmetry superalgebra can be written as follows,

$$\begin{aligned}
 [P_0, Q^\pm] &= 0, & [P_i, Q^+] &= 0, \\
 \{\bar{Q}^+, Q^+\} &= \Gamma^0 P_0, & \{\bar{Q}^-, Q^-\} &= \Gamma^0(P_0 + 2\beta L), \\
 [R, Q^\pm] &= \Gamma_R Q^\pm, & [P_i, Q^-] &= \beta J_{ij} \Gamma^j Q^+, \\
 \{\bar{Q}^-, Q^+\} &= \Gamma^i P_i,
 \end{aligned} \tag{5.33}$$

together with 5.25. In 5.33, R denotes any of the rotation generators R_α or L , and Γ_R is a shorthand for the generator of conventional rotations associated with $R \in SO(4)$, in the corresponding spinor representation of $SO(4)$.

Once we examine the structure of preferred holographic screens in the next subsection, it will be interesting to see how these screens are compatible with the structure of the supersymmetry algebra 5.25 , 5.33 .

5.3.2 Preferred holographic screens

Consider an inertial, comoving observer located at an arbitrary point in space, which we place without any loss of generality at the origin of cartesian coordinates $x_i = 0$. Since we are focusing on the perspective of an observer at the origin, it will be convenient to use either the “bipolar” or the spherical coordinates.

The symmetry arguments that allowed us to identify the preferred screen in Gödel’s universe \mathcal{G}_3 without actually calculating the geodesics can in fact be extended to the supersymmetric solution \mathcal{G}_5 as well. Despite the fact that the full $SO(4)$ rotation symmetry of \mathbf{R}^4 is broken to an $SU(2) \times U(1)$ subgroup, the unbroken group still acts transitively on the three-spheres of constant r . Indeed, one can think of the \mathbf{S}^3 at constant r as a copy of $SU(2)$, on which the full $SO(4)$ rotations would act by the left action of one $SU(2)$ and the right action of the other $SU(2)$. In the Gödel solution, the metric on the \mathbf{S}^3 of constant radius is that of a squashed three-sphere, which still leaves the (transitive) right action by $SU(2)$ unbroken. This unbroken $SU(2)$ is sufficient to reduce our analysis of the location of preferred screens to the maximization of the area of the surfaces \mathbf{S}^3 of constant r as a

function of r (at constant t), precisely as in the simpler case of \mathcal{G}_3 studied in the previous section. Without knowing the precise structure of the null geodesics emitted at some time $t < 0$ in all directions from the origin, the symmetries imply that these geodesics will reach the \mathbf{S}^3 of some fixed radius r at $t = 0$.

Thus, in order to find the preferred holographic screens associated with the inertial comoving observer at the origin, we only need to maximize the volume of the \mathbf{S}^3 at fixed r , as a function of r . The induced metric on the \mathbf{S}^3 of radius r at constant t is given by

$$ds_{\text{ind}}^2 = r^2 d\vartheta^2 + r^2 \cos^2 \vartheta (1 - \beta^2 r^2 \cos^2 \vartheta) d\phi_1^2 + r^2 \sin^2 \vartheta (1 - \beta^2 r^2 \sin^2 \vartheta) d\phi_2^2 - 2\beta^2 r^4 \cos^2 \vartheta \sin^2 \vartheta d\phi_1 d\phi_2, \quad (5.34)$$

implying that the induced area of this surface is given by

$$\mathcal{A}(r) = \int_{\mathbf{S}^3} \sqrt{h_{\text{ind}}} = 2\pi^2 r^3 \sqrt{1 - \beta^2 r^2}, \quad (5.35)$$

where h_{ind} is the determinant of the induced metric 5.34. We conclude that the preferred holographic screen is located at radial distance r (call it $r_{\mathcal{H}}$) where the area 5.35 is maximized,

$$r_{\mathcal{H}} = \frac{\sqrt{3}}{2\beta}. \quad (5.36)$$

The screen carries a Lorentz-signature induced metric,

$$ds_{\mathcal{H}}^2 = -dt^2 - \frac{3}{2\beta} (\cos^2 \vartheta d\phi_1 + \sin^2 \vartheta d\phi_2) dt + \frac{3}{4\beta^2} \left[d\vartheta^2 + \cos^2 \vartheta d\phi_1^2 + \sin^2 \vartheta d\phi_2^2 - \frac{3}{4} (\cos^2 \vartheta d\phi_1 + \sin^2 \vartheta d\phi_2)^2 \right], \quad (5.37)$$

with each spacelike slice of constant t isomorphic to the squashed three-sphere of radius $r_{\mathcal{H}}$ and squashing parameter $3/4$. The screen metric 5.37 seems to exhibit dragging of frames, but this is an artifact of a coordinate choice. Upon introducing new angular coordinates by $\bar{\phi}_1 = \phi_1 - 4\beta t$, $\bar{\phi}_2 = \phi_2 - 4\beta t$, 5.37 becomes

$$ds_{\mathcal{H}}^2 = -4dt^2 + \frac{3}{4\beta^2} \left[d\vartheta^2 + \cos^2 \vartheta d\bar{\phi}_1^2 + \sin^2 \vartheta d\bar{\phi}_2^2 - \frac{3}{4} (\cos^2 \vartheta d\bar{\phi}_1 + \sin^2 \vartheta d\bar{\phi}_2)^2 \right]. \quad (5.38)$$

This phenomenon is analogous to the behavior of horizons in rotating black holes in five dimensions [153] .

The screen and its location in the Gödel universe can be visualized exactly as in Fig. 2, with ϕ now collectively denoting the coordinates on the squashed three-sphere. Again, the preferred screen cuts out a compact region of space with the observer inside, which we will refer to as the holographic region.

The compact preferred holographic screen also implies a finite bound on the entropy that flows through a space-like section of the holographic region. This entropy has to be smaller than one fourth of the area of the screen in Planck units,

$$S \leq \frac{2\pi^3 r_{\mathcal{H}}^3}{\kappa_5^2}. \quad (5.39)$$

(Notice that our κ_5 is related to the 5d Newton constant by $8\pi G_N = \kappa_5^2$.)

It is interesting to analyze the symmetries preserved by the screen. While all the rotation symmetries $SU(2) \times U(1)$ as well as the time translation symmetry are left unbroken, all the space translations are broken by the screen. Similarly, the structure of the supersymmetry algebra reveals that one half of the supercharges (namely Q^-) will be broken by the screen, while the remaining half of supersymmetry represented by Q^+ (and associated with Killing spinors which are simply constant) is compatible with the presence of the screen. Thus, the screen can preserve as much as 1/2 of the full supersymmetry of the Gödel solution, leaving an unbroken symmetry which coincides with the symmetry left unbroken by the choice of the inertial comoving observer. Once we lift the solution to M-theory, we can also think of the preferred comoving observer as a massless particle moving with the speed of light along the extra dimension and preserving 1/2 of supersymmetry. Thus, the symmetries of the observer seem compatible with the symmetries that can be left unbroken by her preferred holographic screen.

In order to verify that this simplified argument for identifying the preferred screens, which relies on the large symmetry of the solution, coincides with the conventional local definition of the screen [143] as the surface of vanishing expansion parameter $\theta = 0$ of the null geodesics emitted from (or, by the time reflection symmetry, sent towards) the origin

in space, we must first analyze the structure of geodesic motion in the Gödel spacetime. This analysis will also refine our understanding of the Gödel universe geometry.

5.3.3 Geodesics in the Gödel universe \mathcal{G}_5

In this subsection we will find all the geodesics in the Gödel universe. First, one can use the symmetries of the solution to simplify the analysis. By homogeneity, it will be sufficient to consider geodesics through the origin P of our coordinate system, $P \equiv \{t = x_m = 0\}$. In any case, for the identification of the preferred screens we are primarily interested in null geodesics emitted from the origin.⁴

We will write the tangent vector to the geodesic as

$$\xi = \dot{t} \frac{\partial}{\partial t} + \dot{r}_1 \frac{\partial}{\partial r_1} + \dot{\phi}_1 \frac{\partial}{\partial \phi_1} + \dot{r}_2 \frac{\partial}{\partial r_2} + \dot{\phi}_2 \frac{\partial}{\partial \phi_2}, \quad (5.40)$$

where $\dot{} \equiv d/d\lambda$ denotes the derivative with respect to an affine parameter λ along the geodesic.

The large amount of symmetry of the Gödel universe allows us to explicitly solve for all the geodesics without any restrictions. First of all, the following integrals of motion will be useful,

$$(\xi, \xi) = -M^2, \quad (\xi, \partial_t) = -E, \quad (5.41)$$

$$(\xi, \partial_{\phi_1}) = L_1, \quad (\xi, \partial_{\phi_2}) = L_2. \quad (5.42)$$

Here L_1, L_2 are the angular momenta in the two preferred planes of rotation. The \pm sign of M^2 corresponds to timelike and spacelike geodesics, with E the energy of the particle in the timelike case. In the null case $M^2 = 0$ we will find it convenient to rescale the affine parameter λ along the geodesic so as to set $E = 1$.

⁴Moreover, since the $SU(2)$ part of the symmetry group acts transitively on the celestial sphere at P , one could rotate the initial momentum vector along the geodesic to lay entirely in the $x_3 = x_4 = 0$ plane. By angular momentum conservation, corresponding to the two Killing vectors $\partial/\partial\phi_1$ and $\partial/\partial\phi_2$, the geodesic would then stay in the $x_3 = x_4 = 0$ plane throughout its history.

The integrals of motion 5.41 imply

$$\dot{\phi}_1 = \beta E + \frac{L_1}{r_1^2}, \quad \dot{\phi}_2 = \beta E + \frac{L_2}{r_2^2}, \quad (5.43)$$

$$\dot{t} = (1 - \beta^2 r_1^2 - \beta^2 r_2^2)E - \beta(L_1 + L_2), \quad (5.44)$$

as well as

$$(\dot{r}_1)^2 + (\dot{r}_2)^2 - (1 - \beta^2 r_1^2 - \beta^2 r_2^2)E^2 + 2\beta E(L_1 + L_2) + \frac{L_1^2}{r_1^2} + \frac{L_2^2}{r_2^2} = -M^2. \quad (5.45)$$

In order to identify the holographic screen we need the null geodesics going through the origin. Note that for non-zero values of the angular momenta L_1 or L_2 , the effective potential for r_1 and r_2 precludes the geodesics from reaching the origin $r_1 = r_2 = 0$. Thus, all the geodesics passing through the origin will have $L_1 = L_2 = 0$, and we focus on those now.⁵ In order to separate \dot{r}_1 from \dot{r}_2 we need one more integral of motion. Consider

$$\begin{aligned} (\xi, R_3) \equiv & (\sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2) \left(\frac{r_2}{r_1} L_1 + \frac{r_1}{r_2} L_2 \right) \\ & + (\sin \phi_1 \cos \phi_2 - \cos \phi_1 \sin \phi_2) (r_2 \dot{r}_1 - r_1 \dot{r}_2). \end{aligned} \quad (5.46)$$

At zero angular momentum, 5.46 has to vanish, implying that the angle ϑ between r_1 and r_2 is another integral of motion. Thus, the equations of motion for the geodesics that pass through the origin of space simplify to

$$(\dot{r})^2 + \beta^2 r^2 E^2 = E^2 - M^2, \quad (5.47)$$

plus 5.43 with L_i set to zero. These can be easily solved, yielding

$$r_1 = \frac{1}{\beta} \sqrt{1 - M^2} \sin(\beta \lambda) \cos \vartheta, \quad (5.48)$$

$$r_2 = \frac{1}{\beta} \sqrt{1 - M^2} \sin(\beta \lambda) \sin \vartheta, \quad (5.49)$$

$$t = \frac{1}{2}(1 + M^2)\lambda + \frac{1}{4\beta}(1 - M^2) \sin(2\beta \lambda) + t_0, \quad (5.50)$$

$$\phi_1 = \beta \lambda + \phi_1^{(0)}, \quad (5.51)$$

$$\phi_2 = \beta \lambda + \phi_2^{(0)}. \quad (5.52)$$

⁵Of course, all the geodesics with nonzero angular momenta can be easily obtained from those with zero angular momenta by the action of the large isometry group of the Gödel metric.

with $\vartheta \in [0, \pi/2)$ and $\phi_1^{(0)}, \phi_2^{(0)} \in [0, 2\pi)$ all constants. We have rescaled the affine parameter λ so as to set E equal to one. For null geodesics, $M^2 = 0$, while for the timelike geodesics $M^2 \in [0, 1]$ as a result of our rescaling of λ . Notice that the comoving time t at the origin (the coordinate corresponding to the Killing vector ∂_t) is *not* a good affine parameter along the null geodesics passing through the origin. Instead, either one of the two main rotation angles ϕ_1, ϕ_2 plays the role of a natural affine parameter (as long as β is nonzero of course).

Even though the spherical coordinate system is not smooth at the origin, it is easy to verify – by switching to the original Cartesian coordinate system – that the system of null geodesics 5.48 represents the complete system of all geodesics passing through the origin. Indeed, the tangent vector to this congruence at $\lambda = 0$ is given in the Cartesian coordinates by

$$\begin{aligned} \xi|_{\lambda=0} = & \frac{\partial}{\partial t} + \cos \vartheta \cos \phi_1^{(0)} \frac{\partial}{\partial x_1} + \cos \vartheta \sin \phi_1^{(0)} \frac{\partial}{\partial x_2} \\ & + \sin \vartheta \cos \phi_2^{(0)} \frac{\partial}{\partial x_3} + \sin \vartheta \sin \phi_2^{(0)} \frac{\partial}{\partial x_4}, \end{aligned} \quad (5.53)$$

demonstrating that the constants $\vartheta, \phi_1^{(0)}$ and $\phi_2^{(0)}$ are indeed parametrizing the entire celestial sphere at the origin.

Thus, we see an interesting refocusing behavior of all geodesics in the Gödel universe: They start moving from the origin towards larger values of r , which at first means larger proper-radius spheres, but then at affine parameter

$$\lambda = \frac{\pi}{2\beta} \quad (5.54)$$

they reach the velocity-of-light surface, located at the largest value r_0 of the radial coordinate r that is accessible by geodesic motion from the origin,

$$r_0 = \frac{1}{\beta}. \quad (5.55)$$

By that time, both ϕ_1 and ϕ_2 change exactly by $\pi/2$. Then it takes another

$$\Delta\lambda = \frac{\pi}{2\beta} \quad (5.56)$$

to complete one period of oscillation and refocus at the origin. The amount of global comoving time coordinate elapsed during the completion of one oscillation cycle equals

$$\Delta t = \frac{\pi}{2\beta}. \quad (5.57)$$

Note that the lightray arrives with its momentum equal to the initial-value momentum; thus, the lightray traveled a full circle in the (x_1, x_2) plane. The same holds true for the (x_3, x_4) plane.

During one refocusing cycle, the proper area of the three-sphere reached by the geodesics reaches a maximum twice, precisely when they reach the preferred screen – first on their way out towards the velocity-of-light surface (where the proper area of the \mathbf{S}^3 goes to zero) and then again on their way back to the origin. In fact, they reach the holographic screen for the first time at affine parameter

$$\lambda = \frac{\pi}{3\beta}, \quad (5.58)$$

one third into the oscillation cycle.

Since any given geodesic moves around a circle in each of the preferred planes of rotation, it is instructive to use the translation symmetries of the solution, and transform 5.48 into the frame associated with the observer at the center of this circular motion. The Killing vectors 5.19 can be easily integrated to give finite translations. For example, we find that a finite translation by a along x_2 is accompanied by an x_1 -dependent translation in t ,

$$x'_2 = x_2 + a, \quad x'_1 = x_1, \quad t' = t + \beta x_1 a. \quad (5.59)$$

When one transforms 5.48 to the primed coordinates associated with the center of the circular motion of a geodesic, the x_1 -dependent time translation 5.59 eliminates the $\sin(2\beta\lambda)$ term in the expression for t as a function of the affine parameter in 5.48. Thus, t becomes a good affine parameter precisely for the class of geodesics that circle around the origin at fixed constant r .

So far, we were mainly concentrating on null geodesics emanating from the origin. The analysis is easily extended to timelike geodesics, which turn out to exhibit a similar cyclic

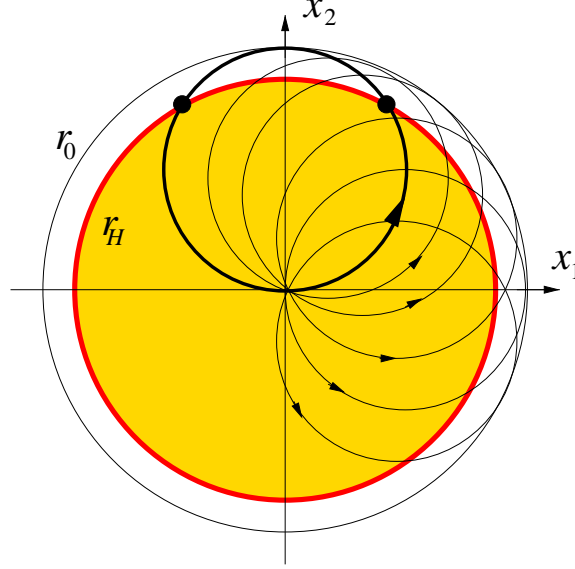


Figure 5.4: The behavior of null geodesics emitted from an arbitrary point P in the Gödel universe, with the initial momentum in the (x_1, x_2) plane, and with several such geodesics indicated. Each geodesic travels along a circular trajectory, reaches the velocity-of-light surface and returns back to P , penetrating the preferred screen exactly twice during each rotation cycle.

behavior. However, they only reach up to a certain critical distance r_M strictly smaller than the distance r_0 of the velocity-of-light surface,

$$r_M = \sqrt{1 - M^2} r_0. \quad (5.60)$$

In terms of the global comoving time coordinate t , the timelike geodesics sent from the origin take longer to refocus at the origin than null geodesics, the refocusing time being

$$\Delta t(M) = \frac{(1 + M^2)\pi}{2\beta}. \quad (5.61)$$

The geodesic expansion θ

We are now in a position to verify that the holographic screen is indeed located at $r_H = \sqrt{3}/2\beta$ by a direct analysis of the geodesics in the Gödel metric. Recall that according

to Bousso's prescription [143], the screen is determined as the surface \mathcal{B} where the geodesic expansion θ vanishes, leading to the “equation of motion” for the preferred holographic screen,

$$\theta = 0, \quad (5.62)$$

with $\theta \equiv h^{\mu\nu} D_\mu \xi_\nu$ defined as the contraction of the covariant derivative $D_\mu \xi_\nu$ of the null covector ξ_μ with respect to the induced metric $h_{\mu\nu}$ on \mathcal{B} .

The null geodesics 5.48 define a congruence whose associated tangent vector is

$$\xi = (1 - \beta^2 r^2) \frac{\partial}{\partial t} + \beta \left(\frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} \right) + \sqrt{1 - \beta^2 r^2} \frac{\partial}{\partial r}. \quad (5.63)$$

Its covector dual (which we denote by the same letter ξ) has a rather simple form,

$$\xi \equiv \xi_\mu dX^\mu = -dt + \sqrt{1 - \beta^2 r^2} dr. \quad (5.64)$$

We can now evaluate the covariant derivative $D_\mu \xi_\nu$ and contract it against the induced metric $h^{\mu\nu}$, to obtain the geodesic expansion θ . After some straightforward algebra,

$$\theta = \frac{3 - 4\beta^2 r^2}{r\sqrt{1 - \beta^2 r^2}}. \quad (5.65)$$

Thus, θ vanishes precisely at $r = r_{\mathcal{H}} \equiv \sqrt{3}/2\beta$, in accord with our anticipation in 5.36. Notice also that θ diverges at the origin and at the velocity-of-light surface, confirming that those are indeed caustics of the geodesic motion.

5.3.4 The Gödel universe of M-theory

The lift of the five-dimensional Gödel universe \mathcal{G}_5 to M-theory involves adding six flat dimensions \mathbf{R}^6 , which we parametrize by coordinates y_a , $a = 1, \dots, 6$. Together, t, x_i and y_a form a coordinate system X^M on \mathbf{R}^{11} , with $M = 0, \dots, 10$. The action of eleven-dimensional supergravity has the form

$$\mathcal{L}_{11} = \frac{1}{2\kappa^2} \int d^{11}X \left(R - \frac{1}{48} G_{MNPQ} G^{MNPQ} + \dots \right), \quad (5.66)$$

where “...” stand for the Chern-Simons term plus fermionic terms. The eleven-dimensional Gödel solution is then given by

$$ds^2 = -(dt + \beta\omega)^2 + \sum_{i=1}^4 dx_i^2 + \sum_{a=1}^6 dy_a^2, \quad (5.67)$$

$$G_{ijab} = 2\beta J_{ij} K_{ab}, \quad (5.68)$$

with all the other non-zero components of G_{MNPQ} related to 5.67 by permutations of indices, and the Kähler form K on the \mathbf{R}^6 factor defined by $K_{12} = -K_{21} = K_{34} = -K_{43} = K_{56} = -K_{65} = 1$.

Consider again the congruence of all null geodesics emitted from the origin in space, where our comoving observer is located. The longitudinal momenta K^a along y_a are conserved, leading to the following congruence of null geodesics:

$$r_1 = \frac{1}{\beta} \sqrt{1 - K^2} \sin(\beta\lambda) \cos \vartheta, \quad (5.69)$$

$$r_2 = \frac{1}{\beta} \sqrt{1 - K^2} \sin(\beta\lambda) \sin \vartheta, \quad (5.70)$$

$$t = \frac{1}{2}(1 + K^2)\lambda + \frac{1}{4\beta}(1 - K^2) \sin(2\beta\lambda) + t_0, \quad (5.71)$$

$$\phi_1 = \beta\lambda + \phi_1^{(0)}, \quad (5.72)$$

$$\phi_2 = \beta\lambda + \phi_2^{(0)}, \quad (5.73)$$

$$y^a = K^a \lambda. \quad (5.74)$$

Just as in the case of four-dimensional Gödel’s solution $\mathcal{G}_3 \times \mathbf{R}$ discussed in the previous section, one can use geodesics in the supersymmetric Gödel solution $\mathcal{G}_5 \times \mathbf{R}^6$ of M-theory to define several different classes of preferred screens. First of all, there is a preferred screen which is a direct product of \mathbf{R}^6 and the screen that we found at $r = r_{\mathcal{H}}$ in \mathcal{G}_5 . This screen is translationally invariant along all the extra dimensions y_a , and clearly satisfies the $\theta = 0$ condition trivially. It is observer-dependent, and should be associated with an observer localized at a point in \mathcal{G}_5 but otherwise delocalized along \mathbf{R}^6 , or with the maximal expansion of lightrays sent with zero momentum K^a from the origin in \mathcal{G}_5 and arbitrary y_a .

In addition, observers localized in a point P both in \mathcal{G}_5 and in \mathbf{R}^6 will naturally see a compact screen in all directions. The precise location of this compact screen can be found by considering the full congruence 5.69 of godesics emitted from P . One can in principle calculate the expansion parameter θ and find the preferred compact screen as the surface of maximal expansion. Using the affine parameter λ and the total momentum $K^2 \equiv K_a K^a$

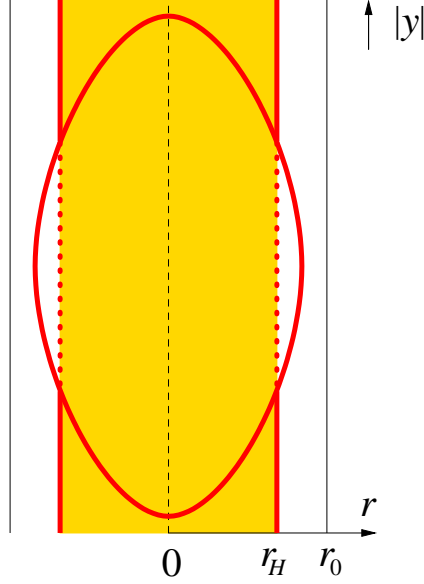


Figure 5.5: The two types of preferred screens in the M-theory Gödel $\mathcal{G}_5 \times \mathbf{R}^6$. The translationally-invariant screen is located at $r_{\mathcal{H}}$ in \mathcal{G}_5 for all values of $|y|$, and can be associated with an extended observer delocalized or wrapped along y_a . The screen associated with a localized observer is compact in all space directions, and extends beyond $r_{\mathcal{H}}$, closer to the velocity-of-light surface r_0 .

along \mathbf{R}^6 as coordinates, the shape of the screen is determined from the $\theta = 0$ condition by a rather complicated implicit function of λ and K^2 ,

$$\begin{aligned}
 0 = \frac{1}{2\lambda} \sin^{-1}(\beta\lambda) \left[(1 - K^2)\beta\lambda \cos(\beta\lambda) + K^2 \sin(\beta\lambda) \right]^{-1} \\
 \times \left[5K^2 + 2(1 - K^2)\beta^2\lambda^2 + (-5K^2 + 4(1 - K^2)\beta^2\lambda^2) \cos(2\beta\lambda) \right. \\
 \left. + 2(3 - K^2)\beta\lambda \sin(2\beta\lambda) \right].
 \end{aligned} \tag{5.75}$$

This screen is compact in all space dimensions, and exhibits $SO(6) \times SU(2) \times U(1)$ rotation invariance, with $SO(6)$ acting on y_a and $SU(2) \times U(1)$ on x_i .

There are several interesting points about this compact screen. First of all, along $y_a = 0$ this screen extends in the r directions beyond the location $r_{\mathcal{H}}$ of the translationally invariant screen. This is in fact intuitively clear: once we add the flat dimensions y_a , the tendency of the geodesics to expand in the y_a dimensions competes against the refocusing behavior of the geodesics in the r direction of \mathcal{G}_5 , effectively slowing down the process of reaching the surface of maximal area, which now happens for a slightly larger value of r . Notice also that the entire compact screen still fits nicely within the velocity-of-light surface as defined by our observer. Therefore, closed timelike curves are again shielded from the observer by this screen.

5.4 Analogies with Holography in de Sitter Space

Holography in de Sitter space is difficult due to the absence of a solvable model or an explicit embedding of de Sitter into string theory. As we have seen in the previous sections, holography in the Gödel universes exhibits notable analogies with holography in de Sitter space.

There are two important classes of preferred holographic screens in de Sitter [143] : First, the future and past infinity are global, observer-independent screens of Euclidean signature. An attempt to formulate holography using these screens [135] has led to the conjectured dS/CFT correspondence [136] . However, it is difficult to associate these global screens with an observer inside de Sitter: Distinct points at future infinity in de Sitter are outside of each other's causal influence, and any operational definition of measurable correlations seems to require a metaobserver.

The second class of screens is more suitable for the description of physics as seen by an observer inside de Sitter [130, 131, 134]: The preferred screen of a given observer is located at his or her cosmological horizon. Since the area of this observer-dependent screen is finite, the strong version of the holographic principle implies a finite number of degrees of freedom

in the quantum mechanics associated with that observer. The finiteness of the number of degrees of freedom accessible to any given observer leads to various conceptual puzzles, such as the recently discussed question of time recurrences [154]. Observers following different trajectories have access to different holographic regions, perhaps suggesting a quantum mechanical description of de Sitter space as a web of infinitely many Hilbert spaces (each associated with an observer and grasping a finite number of degrees of freedom) with a complicated system of maps between them (reflecting the exchange of data between causally connected observers, and the horizon complementarity principle).

Given the conceptual complexity of de Sitter holography, it would be very helpful to have an explicit simple solvable model exhibiting similar properties. We believe that the supersymmetric Gödel universes may provide such a model. Indeed, preferred screens appearing in Gödel holography share many properties with the second type of preferred screens in de Sitter space:

- Both represent an example of homogeneous geometries with screens that are only defined when an observer has been selected. Observers following different worldlines will see different holographic screens.
- The underlying spacetime geometry is homogeneous, but this homogeneity is broken by the selection of the observer, and consequently by the location of the observer-dependent holographic screen, implying that the screen breaks spontaneously some of the symmetries of the naive vacuum. This picture of observer-dependent holography stresses the importance of a local, environmentally-friendly definition of cosmological observables.
- The finite proper area of the holographic screen implies a finite bound on the entropy that flows through the compact holographic region of space associated with the observer. In addition, the strong version of the holographic principle suggests that the observer has only access to a finite number of degrees of freedom. Since the volume of space accessible to the observer is effectively finite, the system has effectively been put

in a finite box. Some of the conceptual difficulties with a possible stringy realization of de Sitter space are connected to the fact that it is very difficult to confine strings in a finite box.

There are also some qualitative differences between Gödel and de Sitter holography worth pointing out:

- In the Gödel universe, the preferred screens are timelike, just as the canonical global screen in AdS space. On the other hand, the observer-dependent preferred screens in de Sitter space are null.
- The Gödel universe is supersymmetric.

In order to decide whether holography in the Gödel universe can be used as a supersymmetric laboratory for exploring conceptual questions arising in de Sitter holography (or more generally, holography in cosmological spacetimes), one needs a more microscopic understanding of the Gödel universes in string and M-theory.

5.5 T-Duality of Gödel Universes

One can compactify one of the flat directions \mathbf{R}^6 (say y_6) of the M-theory Gödel solution on \mathbf{S}^1 with constant radius \mathcal{R} and obtain the following Type IIA Gödel background,

$$ds^2 = -(dt + \beta\omega)^2 + \sum_{i=1}^4 (dx_i)^2 + \sum_{a=1}^5 (dy_a)^2, \quad (5.76)$$

$$H_{ij5} = 2\beta J_{ij}, \quad (5.77)$$

$$F_{ijab} = 2\beta J_{ij} K_{ab}, \quad (5.78)$$

where now in Type IIA theory $a, b \dots = 1, \dots 5$. The dilaton is constant, implying that the string coupling g_s can be kept small everywhere, and the Gödel solution is a solution of weakly coupled Type IIA superstring theory. Now, we can T-dualize along various dimensions.

5.5.1 T-duality to a supersymmetric Type IIB pp-wave

The H -field of the Type IIA Gödel solution 5.76 extends along y_5 , the dimension that was paired up in M-theory with the extra dimension y_6 . It turns out that T-duality along this dimension is particularly interesting. We first rename $y_5 \equiv z$, and use the gauge in which

$$B_{iz} = \beta J_{ij} x^j. \quad (5.79)$$

Due to the absence of $g_{z\mu}$ cross-terms in the metric, no B -field will be generated after T-duality, and one gets

$$ds_{\text{IIB}}^2 = -dt^2 - 2\beta\omega (dt + dz) + \sum_{i=1}^4 dx_i^2 + \sum_{a=1}^4 dy_a^2 + dz^2. \quad (5.80)$$

To see that this Type IIB solution is in fact a supersymmetric pp-wave, it will be convenient to change the coordinates as follows. First, define lightcone coordinates $u = t + z$, $v = t - z$, and also switch from the Cartesian coordinates x_i to the “bipolar” coordinates given in 5.26. Then, we perform a u -dependent rotation in each of the two preferred planes of rotation,

$$\tilde{\phi}_i = \phi_i - \beta u. \quad (5.81)$$

Upon introducing new Cartesian coordinates \tilde{x}_i ,

$$\tilde{x}_1 + i\tilde{x}_2 = r_1 e^{i\tilde{\phi}_1}, \quad (5.82)$$

$$\tilde{x}_3 + i\tilde{x}_4 = r_2 e^{i\tilde{\phi}_2}, \quad (5.83)$$

the Type IIB metric 5.80 T-dual to the Gödel universe becomes

$$ds_{\text{IIB}}^2 = -du dv - \beta^2 \left(\sum_{i=1}^4 \tilde{x}_i^2 \right) du^2 + \sum_{i=1}^4 d\tilde{x}_i^2 + \sum_{a=1}^4 dy_a^2. \quad (5.84)$$

This metric has the standard form of a supersymmetric pp-wave, with the Gödel rotation parameter β precisely equal to the conventionally normalized μ parameter of the pp-wave. One can also easily T-dualize the Ramond-Ramond fields: The self-dual Type IIB five-form of the Type IIB solution is given by $F_5 \sim du \wedge \tilde{J} \wedge K$, where $\tilde{J} = \sum_{i,j=1}^4 J_{ij} d\tilde{x}_i \wedge d\tilde{x}_j$ and

$K = \sum_{a,b=1}^4 K_{ab} dy_a \wedge dy_b$. This Type IIB solution is in fact the supersymmetric pp-wave resulting from the Penrose limit of the near-horizon $AdS_3 \times \mathbf{S}^3 \times T^4$ geometry of a system of intersecting D3-branes, and was first found in [155].

5.5.2 Gödel/pp-wave T-duality

We have shown that the Type IIA Gödel universe is T-dual to a Type IIB pp-wave. One can turn this observation around, and ask whether other known pp-waves can also be T-dual to new Gödel-like universes. We indeed find a rich picture of Gödel/pp-wave duality which goes beyond the scope of the pp-wave T-dualities discussed in the literature (see, e.g., [156]).

Before generalizing the result of the previous subsection to a broader class of Gödel/pp-wave pairs, it is instructive to first clarify which Killing dimension of the Type IIB pp-wave is being compactified on \mathbf{S}^1 and T-dualized to produce the Type IIA Gödel universe. Consider first the Killing vector

$$\xi_0 = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \quad (5.85)$$

of the Type IIB pp-wave background. This vector is space-like at the origin, but becomes time-like at some critical radial distance. One can remedy this problem by augmenting ξ_0 with a rotation in each of the two preferred planes

$$\xi = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} + \beta \left(\frac{\partial}{\partial \tilde{\phi}_1} + \frac{\partial}{\partial \tilde{\phi}_2} \right). \quad (5.86)$$

This Killing vector ξ is everywhere spacelike, with the space-like rotation off-setting the effect of the du^2 terms in the metric to keep this modified Killing vector spacelike. Moreover, the norm of ξ is

$$|\xi|^2 = 1. \quad (5.87)$$

Consequently, if we compactify the orbit of ξ on a circle of fixed radius R and T-dualize, the dilaton field of the resulting solution will stay constant. This T-duality is precisely the inverse of the IIA \rightarrow IIB T-duality that maps the Gödel solution to the pp-wave. Note that

closed timelike curves are introduced even though the orbifold action is generated by an everywhere-spacelike Killing vector.

As was pointed out to us after the completion of this work, the T-duality relation between the 5d Gödel universe 5.76 and the geometry of 5.80 can also be obtained from a T-duality relation found a week earlier by Herdeiro in [160] , by taking the limit of zero charges $P = Q = Q_{KK} = 0$ in Eqn. (4.11) of [160].⁶ However, it was not realized in [160] that the T-dual of the Gödel universe is a supersymmetric Hpp-wave; instead, this T-dual was conjecturally interpreted in [160] as a rotating background.

5.5.3 New supersymmetric Gödel universes in string and M-theory

These observations lead to a very simple and general prescription for constructing a large class of Gödel/pp-wave T-dual pairs. Start with any pp-wave in which an analog of the Killing vector ξ of 5.86 (and satisfying 5.87 if we want constant g_s) can be identified. Compactification on \mathbf{S}^1 along this Killing direction followed by T-duality produces a Gödel like solution of the T-dual string theory.

As an example of this, we present a new supersymmetric Gödel universe of Type IIA theory, as the T-dual of the maximally supersymmetric Type IIB pp-wave [157] . Using the obvious generalization of 5.86 that now involves four independent rotations in four independent two-planes of the pp-wave, we obtain a Type IIA geometry with a constant H_3 and F_4 . This Type IIA solution can be lifted to an M-theory solution of \mathbf{R}^{11} topology. Its metric factorizes to a product of a non-trivial metric on a \mathcal{G}_9 factor and the flat metric on \mathbf{R}^2 ,

$$ds^2 = -(dt + \beta\varpi)^2 + \sum_{I=1}^8 (dx_I)^2 + \sum_{A=1}^2 (dy_A)^2, \quad (5.88)$$

$$\varpi = J_{IJ}x_I dx_J, \quad (5.89)$$

⁶We thank Harvey Reall for pointing this out to us.

and the four-form strength can be written as

$$G_{ijkl} = 4\beta\epsilon_{ijkl}, \quad G_{ijAB} = -2\beta J_{ij}K_{AB}, \quad (5.90)$$

$$G_{mnpq} = -4\beta\epsilon_{mnpq}, \quad G_{mnAB} = -2\beta J_{mn}K_{AB}, \quad (5.91)$$

where $i, \dots = 1, \dots, 4$ and $m, \dots = 5, \dots, 8$, while the indices $I, \dots = 1, \dots, 8$ and $A, B = 1, 2$; all the non-zero components of the Kähler forms J_{IJ} and K_{AB} are now given by $J_{12} = -J_{21} = J_{34} = -J_{43} = J_{56} = -J_{65} = J_{78} = -J_{87} = 1$ and $K_{12} = -K_{21} = 1$.

This new supersymmetric Gödel solution $\mathcal{G}_9 \times \mathbf{R}^2$ of M-theory exhibits exactly the same qualitative holographic features as the $\mathcal{G}_5 \times \mathbf{R}^6$ solution. In particular one finds compact closed timelike curves that are topologically large, and the analysis of geodesics reveals the same qualitative structure of holographic screens.

5.6 Geometry of the Gödel Universes

In this section, we collect various aspects of the semi-Riemannian geometry of the Gödel universes \mathcal{G}_5 and \mathcal{G}_9 that play a central role in this chapter. We are using the $+++$ conventions of MTW [158] ; in particular, our metric is of the “mostly plus” signature.

5.6.1 The five-dimensional Gödel universe

In the original Cartesian coordinates t, x_i it is natural to introduce a vielbein

$$e^0 = dt + \beta\omega, \quad e^i = dx_i, \quad i = 1, \dots, 4, \quad (5.92)$$

so that the metric on \mathcal{G}_5 can be written simply as

$$g_{\mu\nu} = -e_\mu^0 e_\nu^0 + \sum_i e_\mu^i e_\nu^i. \quad (5.93)$$

In this vielbein, the spin connection one-forms are

$$\Omega_{ij} = \beta J_{ij} dt + \beta^2 J_{ij} J_{kl} x_k dx_\ell, \quad (5.94)$$

$$\Omega_{i0} = -\Omega_{0i} = \beta J_{ij} dx_j. \quad (5.95)$$

These simple expressions for the spin connection can be used to easily extract the form of the Ricci tensor in the Cartesian coordinates,

$$R_{\mu\nu}dX^\mu dX^\nu = 4\beta^2 dt^2 + 8\beta^3 J_{ij}x_i dt dx_j + 2\beta^2 (\delta_{ij} - 2\beta^2 J_{ik}J_{j\ell}x_k x_\ell) dx_i dx_j. \quad (5.96)$$

The scalar curvature is constant,

$$R = 4\beta^2, \quad (5.97)$$

as is indeed implied by the homogeneity of the solution. The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ has the pressureless fluid form,

$$G_{\mu\nu}dX^\mu dX^\nu = 6\beta^2 dt^2 + 12\beta^3 J_{ij}x_i dx_j dt + 6\beta^4 J_{ik}J_{j\ell}x_k x_\ell dx_i dx_j \quad (5.98)$$

$$= 6\beta^2 u_\mu u_\nu dX^\mu dX^\nu, \quad (5.99)$$

with

$$u_\mu dX^\mu = -dt - \beta J_{ij}x_i dx_j \quad (5.100)$$

the covariant dual of the timelike Killing vector $\partial/\partial t$. This is matched by the energy-momentum tensor of the constant gauge field strength $F \sim J$, which is also of the pressureless fluid form.

For the calculation of the geodesic expansion parameter θ in the body of the chapter, it is also useful to know the non-zero Christoffel symbols in the “bipolar” coordinates $(r_1, \phi_1, r_2, \phi_2)$,

$$\Gamma_{r_1 t}^t = \beta^2 r_1, \quad \Gamma_{\phi_1 t}^{r_1} = \beta r_1, \quad \Gamma_{r_1 t}^{\phi_1} = -\frac{\beta}{r_1}, \quad (5.101)$$

$$\Gamma_{\phi_1 r_1}^t = \beta^3 r_1^3, \quad \Gamma_{\phi_1 \phi_1}^{r_1} = -r_1(1 - 2\beta^2 r_1^2), \quad \Gamma_{\phi_1 r_1}^{\phi_1} = \frac{1}{r_1} - \beta^2 r_1, \quad (5.102)$$

$$\Gamma_{\phi_2 r_1}^t = \beta^3 r_1 r_2^2, \quad \Gamma_{\phi_1 \phi_2}^{r_1} = \beta^2 r_1 r_2^2, \quad \Gamma_{\phi_2 r_1}^{\phi_1} = -\frac{\beta^2 r_2^2}{r_1}, \quad (5.103)$$

$$\Gamma_{r_2 t}^t = \beta^2 r_2, \quad \Gamma_{\phi_2 t}^{r_2} = \beta r_2, \quad \Gamma_{r_2 t}^{\phi_2} = -\frac{\beta}{r_2}, \quad (5.104)$$

$$\Gamma_{\phi_2 r_2}^t = \beta^3 r_2^3, \quad \Gamma_{\phi_2 \phi_2}^{r_2} = -r_2(1 - 2\beta^2 r_2^2), \quad \Gamma_{\phi_2 r_2}^{\phi_2} = \frac{1}{r_2} - \beta^2 r_2, \quad (5.105)$$

$$\Gamma_{\phi_1 r_2}^t = \beta^3 r_2 r_1^2, \quad \Gamma_{\phi_1 \phi_2}^{r_2} = \beta^2 r_1^2 r_2, \quad \Gamma_{\phi_1 r_2}^{\phi_2} = -\frac{\beta^2 r_1^2}{r_2}. \quad (5.106)$$

5.6.2 The nine-dimensional Gödel universe

This solution, discussed in Section 5, is T-dual to the maximally supersymmetric Type IIB pp-wave. We again introduce the natural vielbein in which the metric is of the form 5.93,

$$e^0 = dt + \beta\omega, \quad e^I = dx_I, \quad i = 1, \dots, 8. \quad (5.107)$$

In this basis, the spin connection one-forms are given by

$$\Omega_{IJ} = \beta J_{IJ} dt + \beta^2 J_{IJ} J_{KL} x_K dx_L, \quad (5.108)$$

$$\Omega_{I0} = -\Omega_{0I} = \beta J_{IJ} dx_J, \quad (5.109)$$

with the Ricci tensor

$$\begin{aligned} R_{MN} dX^M dX^N &= 8\beta^2 dt^2 + 16\beta^3 J_{IJ} x_I dx_J dt \\ &\quad + (2\beta^2 \delta_{IJ} + 8\beta^4 J_{IK} x_K J_{JL} x_L) dx_I dx_J, \end{aligned} \quad (5.110)$$

the scalar curvature

$$R = 8\beta^2, \quad (5.111)$$

and the Einstein tensor

$$\begin{aligned} (R_{MN} - \frac{1}{2} R g_{MN}) dX^M dx^N &= 12\beta^2 dt^2 + 24\beta^3 J_{IJ} x_I dx_J dt \\ &\quad - (2\beta^2 \delta_{IJ} - 12\beta^4 J_{IK} x_K J_{JL} x_L) dx_I dx_J. \end{aligned} \quad (5.112)$$

Notice that unlike in the case of the five-dimensional Gödel solution, the Einstein tensor of the nine-dimensional Gödel universe is no longer of the pressureless fluid form.

5.7 Discussion

Following a phenomenological approach to holography, we have identified preferred holographic screens as seen by inertial observers in a class of homogeneous universes of the Gödel type, with closed timelike curves. The structure of holographic screens change dramatically the question of causality, by hiding all closed timelike curves or breaking them into causal

pieces. It is tempting to suspect that holography serves as the chronology protection agency, and in combination with a version of the complementarity principle can lead to a consistent quantum mechanical description of this universe. We also noticed close analogies with the structure of holographic screens in de Sitter space, which can make the Gödel universes an interesting supersymmetric laboratory for exploring de Sitter holography. This phenomenological identification of natural screens does not tell us, however, whether the holographic dual is given by some self-consistent quantum mechanics, or whether the pathology of closed timelike curves is just translated into some inconsistency of the holographic dual. These and similar questions require a microscopic understanding of holography in Gödel universes in string or M-theory. We have found evidence that the Gödel-like cosmologies represent a remarkable and highly solvable class of solutions of string theory, and are in fact T-dual to solvable supersymmetric pp-wave solutions. Further investigation of microscopic aspects of Gödel universes and their holography in string and M-theory is in progress [145].

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